Riemann-Lebesgue lemma:

$$\lim_{n\to\infty} \int_a^b f(x)\sin(nx)dx = 0.$$

Co-area formula:

$$\int_{\mathbb{R}^n} f dx = \int_0^\infty \int_{\partial B_r} f \, d\sigma dr.$$

If

$$|f(x)| \le C(1+|x|)^{-\gamma}$$

then f is integrable if $\gamma > n$.

Calculus problems

1. Find the values of γ such that the double sum converges

$$\sum_{m,n\ge 1} \frac{1}{(m+\sqrt{n})^\gamma} < \infty.$$

2. Show that the limit exists

$$\lim_{a\to\infty} \int_0^a \sin x \, \sin x^2 \, dx.$$

3. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \ge 0$ for all real x. Prove that $|f(x)|$ is bounded.

4. Let $f(t) = \sum_{j=1}^N a_j \sin(2\pi jt)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeroes (including multiplicities) of $\frac{d^k f}{dt^k}$. Prove that

$$N_0 \le N_1 \le N_2 \le \cdots \text{ and } \lim_{k\to\infty} N_k = 2N.$$

5. Let f be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x,y,z)$ over the surface of S equals 0. Must $f(x,y,z)$ be identically 0?

6. Let $a_i, b_i \neq 0$ such that

$$\sum_{i=1}^{n} a_i |\sin(b_i x + i)| \geq 0.$$

Show that $\sum a_i \geq 0$.

7. Let $F = (u, v)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives u_x, u_y, v_x, v_y that are positive everywhere. Suppose that

$$u_x v_y - \frac{1}{4}(u_y + v_x)^2 > 0.$$

Prove that F is one-to-one.

8. Let

$$I(R) = \iint_{x^2+y^2 \leq R^2} \left(\frac{1 + 2x^2}{1 + x^4 + 6x^2y^2 + y^4} - \frac{1 + y^2}{2 + x^4 + y^4} \right) dx\, dy.$$

Find

$$\lim_{R \to \infty} I(R),$$

or show that this limit does not exist.

9. Let $y_1, .., y_n$, $z_1, .., z_n$ be points in the plane with different centers of mass

$$\frac{1}{n} \sum y_k \neq \frac{1}{n} \sum z_k.$$

Show that there are infinitely many x's such that

$$\sum |x - y_k| = \sum |x - z_k|.$$

10. Let $f > 0$ be continuous on $[0, 1]$, and let $x_1 < x_2 < .. < x_n = 1$ be such that

$$\int_0^{x_1} f(x)dx = \int_{x_1}^{x_2} f(x)dx = ... = \int_{x_{n-1}}^{x_n} f(x)dx.$$

Compute

$$\lim_{n \to \infty} \frac{x_1 + x_2 + .. + x_n}{n}.$$

11. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.