Recap of ~~Spring~~ Fall Seminar on Orbit method

DEF $G = PGL_2(\mathbb{R})$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) = \mathrm{sp}(A, B, C)$

where $A = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $B = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $C = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

$\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$, $\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}$
$\mathfrak{g}^\wedge = \mathrm{Hom}(\mathfrak{g}, \mathbb{C}^{(1)})$, $i\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, i\mathbb{R})$

Prop > $\mathfrak{g}^\wedge \cong i\mathfrak{g}^*$ by exponentiation.
> $\mathfrak{g} \cong \mathfrak{g}^*$ by trace pairing (so we will often confuse them)
$X \mapsto \mathrm{tr}(X\,\_)$
($\frac{1}{2}A \leftrightarrow \underset{\text{(dual basis)}}{A^*}$ since $\mathrm{tr}(\frac{1}{2}A \cdot A) = 1$)

DEF Coadjoint action by G on $\mathfrak{g}^*_{\mathbb{C}}$,
$X \in \mathfrak{g} \cong \mathfrak{g}^*$, $g \in G$
$g \cdot X = gXg^{-1}$

DEF: Orbits under coadjoint action

$\mathbb{R}^3 \overset{\text{bij}}{\cong} \mathfrak{g}^*$
$(a, b, c) \to aA + bB + cC$

Coadj action preserves det.

Claim: Orbits : (i) $)($: $\mathcal{O}_{TA}$ $T > 0$
$\{a^2 + b^2 - c^2 = T\}$
(ii) $\substack{\smile \\ \frown}$: $\mathcal{O}_{TC}$ $T > 0$, $\{a^2 + b^2 - c^2 = -T\}$
(iii) $X \cup \bullet$: $\mathcal{O}_{B+C} \cup \mathcal{O}_0$ (det 0)
(iv)

DEF: GIT quotient : Further identifying the orbits of $sl_2(\mathbb{R})^*$ by the determinant. (hence $[\Theta_{B-C}] = [\theta_0]$)

Denoted $[g^*_{\mathbb{C}}]$

Properties

> GIT quotients can be defined more generally.
  But not needed in this case

> Have manifold structure. & can do computations.
  Symplectic measure ω_θ on Θ.

> Irred unitary tempered aut rep of $PGL_2(\mathbb{R})$ $\longrightarrow$ GIT quotient

$$\pi \mapsto \lambda_\pi$$

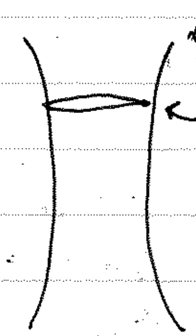
$$P(iT) \longrightarrow \Theta_{iTA} \subseteq g^*_{\mathbb{C}}$$

$$\mathcal{D}^{\pm}(k) \longrightarrow \Theta_{1/2(k-1)iC} \subseteq g^*_{\mathbb{C}}$$

(Computation done last semester)

> Decomposition of $G = PGL_2(\mathbb{R})$ reps π into reps σ of $H = SO(2)$
$\longleftrightarrow$ Decomposition of Θ_π along Θ_σ
(horizontal slices)

Θ_π (principal series)



$\leftarrow \Theta_\sigma$ (Weight σ vector)

> Scaling & conjugating on GIT quotients.

- Scaling on $\mathfrak{g}^*_{\mathbb{C}}$ $\Rightarrow$ Scaling on GIT & Conj.
- Conjugating on $\mathfrak{g}^*_{\mathbb{C}}$

> Eigenvalues : $[M]$ be a GIT quotient (orbit of $M \in \mathfrak{g}$)

$\rightsquigarrow$ Associated polynomial $P_\lambda(x)$ of degree 2

$$P_{[M]}(X) = X^2 - \operatorname{tr}(M)X + \det M = X^2 + \det M$$

$$\text{Eigenvalues} = \text{Roots of } P_{[M]}(X) = \{\lambda_1, \lambda_2\} = ev([M])$$

Scaling $[M]$ by $r_1 \in \mathbb{C}$ $\Rightarrow$ Scaling $\{\lambda_1, \lambda_2\}$ to $\{r_1\lambda_1, r_1\lambda_2\}$

Conj. $[M]$ $\Rightarrow$ Conj. $\{\lambda_1, \lambda_2\}$ to $\{\bar{\lambda}_1, \bar{\lambda}_2\}$

> GIT quotient of $[\mathfrak{g}^\wedge]$ ($\subseteq \mathfrak{g}^*_{\mathbb{C}}$)
$= \{ [M] \mid ev([M]) = ev([-\bar{M}]) \}$

> π irred tempered aut rep, π unitary $\Rightarrow$ $\lambda_\pi \subseteq [\mathfrak{g}^\wedge]$
eigval $\{\lambda_{\pi,1}, \dots, \lambda_{\pi,n}\}$

(Almost (up to 1), same as satake parameters)

Prop : π irred rep of G,

$$T^{-1}\lambda_\pi \in \Omega \quad \Leftrightarrow \quad |\lambda_{\pi,i}| = O(T)$$

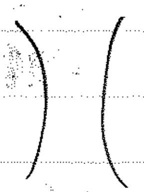
fixed compact subset of $[\mathfrak{g}^*_{\mathbb{C}}]$ — eigval

$\Leftrightarrow$ Langlands parameters $= O(T)$

Symplectic measure on $\mathcal{O}_\pi$ in $PGL_2(\mathbb{R})$ case.

$\pi = P(iT)$ Principal series. $\quad \Delta$ eigval $= \frac{1}{4} + T^2$

$$\mathcal{O}_\pi = \{(a,b,c) \in \mathbb{R}^3 \mid a^2 + b^2 - c^2 = T^2\} = \mathcal{O}_{TA}$$



Parametrised by $\tau(\phi, v)$
$= T(\cosh v \cos\phi, \cosh v \sin\phi, \sinh v)$

$v \in \mathbb{R}, \quad \phi \in [0, 2\pi)$

$$\omega_{\mathcal{O}} = \frac{T\cosh v}{4\pi}\, dv \wedge d\phi$$

Kirillov's Formula

Distribution character : (π, V) irred unitary rep of G.
(H.C.)

$f \in C_c^\infty(G)$, $\quad \pi(f)v = \int_G \pi(g)v\, f(g)\, dg$.
$v \in V$. $\quad$ Conv. op.

$\exists\, \chi_\pi : G \to \mathbb{C}$ st $\quad \mathrm{tr}(\pi(f)) = \int_G \chi_\pi(g) f(g)\, dg$

χ_π determines the rep.

Kirillov: Values of χ_π $\leftrightsquigarrow$ Integral in $\mathcal{O}_\pi$

for π tempered, unitary, irrep of G

Thm: $X \in \mathfrak{g}$ in a small nbh around 0.

$$\chi_\pi(\exp(X)) = \frac{1}{\sqrt{j(X)}} \underbrace{\int_{\xi \in \mathcal{O}_\pi} e^{\langle \xi, X\rangle} \, \omega_{\mathcal{O}_\pi}(\xi)}_{\text{FT of symplectic measure.}}$$

$$j(X) = \mathrm{Jac}(\exp : \mathfrak{g} \to G)$$

End goal: Bounding Matrix coeff integrals.

Need to bound: (i) $\pi_T = \rho(iT)$

$$\sum_{v \in OB(\pi_T)} Q(\pi(f_T)v) \gg T^{-1/2}$$

where: (i) $f \in C_c^\infty(G)$ chosen nicely.
(as in Kuznetsov / Relative TF)

(ii) $Q(v) = \int_K \langle \pi(k)v, v\rangle \, dk$

These appear in RTF $\left(G = PGL_n(\mathbb{R}), \ K = SO(n)\right)$

Orbit method gives us a good choice of f.

(ii) Geometric side of RTF

Choice of f using $Op(a)$

Need : > $\dot{\pi}(f) = Op(a)$ to be a proj. operator over short families.

(averaging over short families is always difficult in applications of KTF)

Then, Size of family $= Tr(Op(a))$

This is calculated using Orbit method.

> Hence we need : $Op(a) \cdot Op(a) = Op(a)$

where a = smooth Bump function on $\mathfrak{g}^\wedge$

~~Defining Op(a)~~

Actually, $Im(Op(a))$ are microlocalised vectors.

"Localised nicely in both freq & time"

$Xv \approx i\langle \tau, X\rangle v$ For fixed τ.

$X = A, B, C.$

Approx eigvect in all directions.

Microlocal vectors are important in sub-convexity estimates.

(See Nelson : Some exercises concerning localised vectors in low rank)