Fall 2022 Directed Reading Program Colloquium

Martingale Proof of Kolmogorov's $0-1$ Law and the Strong Law of Large Numbers

Krystal Sun

Kolmogorov's $0-1$ Law and the Strong Law of Large Numbers are two important theorems in classical probability. The first one states that the tail σ-algebra of a sequence of independent random variables is trivial, and the second states that the average of an independent, identically distributed sequence of integrable random variables converges to their mean almost surely. In this presentation, I will show the martingale proof of these two theorems.

The Yoneda Lemma and the Presheaf Category

Tony Xiao

Given a (locally small) category $\mathcal{C}$ and $c \in \mathcal{C}$, the Yoneda lemma states that for any F in the functor category $Funct(C^{op}, Set)$, we have

$$Nat(h^c, F) \cong F(c)$$

where $Nat(h^c, F)$ is the set of natural transformation from the representable functor h^c to F. In particular, the Yoneda lemma implies that the Yoneda embedding is fully faithful. We will first introduce these notions, including functor categories, natural transformations, representable functors, the Yoneda embedding, etc. Next, we will present an intuitive understanding of the Yoneda lemma. If time permits, we will discuss the implications of the Yoneda lemma in the presheaf category.

The Fredholm Alternative and Applications

Heyuan Yao

The Fredholm alternative is an important theorem in functional analysis. In this talk, I will start with a basic linear algebra result and introduce the definitions of Banch spaces and compact operators. Then I will give the theorem statement and show how the Fredholm alternative helps to prove the existence and uniqueness of the weak solution to Poisson's equation $-\Delta u = f$ where $u \in W_0^{1,2}(D), f \in \mathcal{L}^2(D)$ and $D \subset \mathbb{R}^n$ is open and bounded.

The Markov Property for Brownian Motion

Rizwan Kazi

In this presentation, we will prove the Markov property for one dimensional and multidimensional Brownian motion. Brownian motion describes the random motion of a particle in a fluid. Brownian motion also has a Markov property, which means that its future behavior is determined by its current state, rather than its history. After proving the Markov property of Brownian motion, we will discuss applications to physics and finance.

Reciprocity Laws

David Chen

There is a long chain of reciprocity laws in number theory. We will present results in class field theory relevant to these results. In particular, we will sketch a proof of the local reciprocity law and mention the global analogues.

Statistical Mechanics Helps Us Count Alternating Sign Matrices

Noah Bergam

The enumeration of alternating sign matrices (ASMs) presents a profound, deceptively simple combinatorial problem, with wide-ranging connections across mathematics. These objects have a particularly interesting correspondence with statistical mechanics and, in particular, a lattice model called "square ice." In this presentation, we motivate the study of ASMs and sketch Kuperberg's 1995 proof of the ASM Conjecture, which uses prominent tools of statistical mechanics such as the Yang-Baxter Equation.

Shiv Yajnik

Khovanov's Categorification of the Jones Polynomial

One of the aims of knot theorists has been to discover a consistent method of classifying knots. Khovanov's categorification of the Jones Polynomial, known as Khovanov homology, offers an step forward towards these goals and provides insight about surface maps between knots (cobordisms). In this presentation, I will introduce some basic definitions in knot theory and then explain some computations in Khovanov homology.

Introdution to Elliptic Curves and Mordell's Theorem

Jennifer Luo

Elliptic curves have played an important role in algebra, number theory, and geometry. We will introduce the concept and basic properties of elliptic curves and emphasize one of the fundamental theorems in the subject, Mordell's theorem.

Sparse Linear Models in High Dimensions

Ekene Ezuenala

Linear models are widely used in statistics. In low-dimensional settings where the number of predictors is much less than the sample size, the theory of linear models is classical. This semester, I learned about the theory of linear models in high-dimensional settings. In these settings, to work with the model, we impose additional structure on the unknown vector of regression coefficients. I will present a noiseless linear model in high dimensions.

A Graph Theoretic Approach to the Art Gallery Problem

Keila Leonard

The Art Gallery Problem is a problem in computational geometry first posited in 1973 by Victor Klee. The problem considers the question of how many guards would be needed to guard a space represented by a polygon. In this talk, I will present an alternate proof of this problem concerning convex polygons using the language and methods of graph theory.

The Picard Group

Stanley Jian

I will define holomorphic line bundles over a complex manifold, and explain how the isomorphism classes of these line bundles form a group under the $\otimes$ operation. This is the Picard group. If time permits, I will present some results about the Picard group.