ASYMPTOTIC STABILITIES IN ALGEBRA AND GEOMETRY

PROJECT LEADER : QUENTIN GUIGNARD

1. **Introduction.** The word **stability**, used in the context of algebra or topology, refers to the emergence of asymptotic patterns in a sequence of objects $(V_n)_n$. For example, one might seek such patterns in the series of symmetric groups $\mathfrak{S}_n$ when n goes to infinity, starting with the observation that conjugacy classes of $\mathfrak{S}_n$, or its irreducible representations, are parametrized independently of n by Young tableaux. Subtler stabilities emerge by considering more interesting invariants of $\mathfrak{S}_n$ such as its homology. This leads to our first example of stability:

(1) **Homological Stability**. It was proved by Nakaoka in the early '60s that for each integer q the homomorphism

$$H_q(\mathfrak{S}_n) \to H_q(\mathfrak{S}_{n+1}),$$

between homology groups is an isomorphism for n large enough (depending on q). This has been generalized by various authors to sequences of classical groups, such as GL_n, to sequences of mapping class groups, to groups of automorphisms of free groups, etc. One can also generalize in another direction by allowing homology with coefficients in a sequence of representations. In yet another research direction, one might try to quantify in terms of q what is meant by "n large enough" in the stability statement; for example, Nakaoka obtained that $n \geq 2q + 1$ is enough for symmetric groups.

In topology, we have finer invariants than homology, namely homotopy groups. This yields a second example of stability:

(2) **Homotopical Stability**. We consider the sequence S_n of higher dimensional spheres. Freudenthal proved in the '30s that for any integer q, the homomorphisms

$$\pi_{n+q}(S_n) \to \pi_{n+1+q}(S_{n+1}),$$

are isomorphisms for n large enough (depending on q). The limit groups have been computed for $q \leq 64$ or so, but not in general. Freudenthal actually proved a more general statement regarding successive supensions of topological spaces, a pioneering result in the subsequent development of stable homotopy theory.

Recent developements in stability result were provided by the recent theory of FI-modules by Church-Ellenberg-Farb. Their theory leads to our third example of stability:

(3) **Representation Stability**. Let M be a connected oriented manifold and let $\mathrm{Conf}_n(M)$ be the manifold parametrizing ordered n-uples of distinct points on M. Then for each integer q, the sequence

$$H^q(\mathrm{Conf}_n(M), \mathbb{Q}),$$

of cohomology groups, stabilizes (in an appropriated sense) as a sequence of representations of $\mathfrak{S}_n$. This extends to many naturally occurring sequences of representations of $\mathfrak{S}_n$.

2. **Project Outline.** We will not consider homotopical stability, nor any kind of algebraic topology. We will instead focus on homological and representation stability.

(1) The first task will be to define group homology and to understand Nakaoka's stability, i.e. homological stabilisation for symmetric groups. This will be a perfect occasion to learn about spectral sequences and their use.
(2) We will then consider homological stability for GL_n. The goal is arrive at the realization that the proof is nearly identical to the one for $\mathfrak{S}_n$, and thus to give a recipe to obtain other homological stabilities.
(3) We will then move on to FI-modules. The goal is to understand the main results of Church-Ellenberg-Farb and how to apply them in concrete situations.
(4) For the last part of the project, the students will be invited to generalize the results of (3) to a setting designed to obtain representation stability over GL_n, by drawing inspiration from the analogy between (1) and (2).

3. **Prerequisites.** Students should be well acquainted with basic commutative algebra (e.g. modules, finiteness, noetherianity) and representations of finite groups. It is preferable to have some basic knowledge of the language of categories and of homological algebra; however group (co)homology and spectral sequences are not prerequisites and will be reviewed during the project.