Last passage percolation on mean-field graphs

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Last passage percolation: Last passage percolation (LPP) is a class of probabilistic models with connections to statistical physics. The one which has received the most study from the mathematics community is defined on $\mathbb{Z}^2$, and we describe it now.

To each vertex v of $\mathbb{Z}^2$, we associate an independent non-negative random variable ξ_v drawn from a common fixed probability distribution. You can think of these random numbers as rewards or weights. Next, we consider nearest-neighbor up-right paths in $\mathbb{Z}^2$ between two fixed points: any such path can move only up or right by one unit at each step. A path also has a weight associated to it, which is obtained by adding up the random variables associated to every vertex which lies along it.

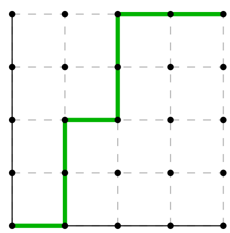

Figure 1: Paths in LPP on $\mathbb{Z}^2$.

Let us fix two points, $(1,1)$ and (n,n). Now, there are many up-right paths which connect these two points, and each has a weight associated to it. A typical object of interest is the *maximum* weight M_n over all these paths. How fast does M_n grow as n increases, on average? It's a random object, so what size are its fluctuations around its average, and can we understand the probabilistic properties of the fluctuations?

Progress has been made on answering these questions for particular cases of the distribution of the ξ_v, but not much is known for any other distribution (at least for the second question). This is in spite of over 20 years of work!

In this project, we will simplify the problem. Instead of working on $\mathbb{Z}^2$, we will consider analogous models on simpler graphs with less geometric structure, where it may be easier to consider similar questions. This is a common strategy in statistical physics—such simpler models which remove geometric structure are often called *mean-field* models. And indeed, there is some previous work on mean-field LPP models.

Some examples of the graphs we may consider are the complete graph on n vertices and the Erdős-Rényi graph $G(n,p)$. The latter graph is a random graph: we start with n vertices and, for every pair of vertices, we connect them with an edge with probability p, independent of everything else. The large degree of independence and the consequent lack of rigid geometry (as is present in graphs like $\mathbb{Z}^2$) often makes this a more tractable arena.

We will be interested in a number of aspects of LPP on these graphs. These include the scale of the random fluctuations of the last passage value on a heuristic level and showing that its variance is sublinear (i.e., the variance in the graph of size n grows at a rate slower than n); the behavior of the random path which achieves the maximum weight; estimates of probabilities of seeing last passage values which are unusually large or small on the first order scale, i.e., on a scale much larger than typical fluctuations (in probability jargon, large deviations). We will see connections to other areas of mathematics as well; for instance, recent work from 2017 of Duminil-Copin–Raoufi–Tassion provides a method of proving sublinear variance by finding an efficient randomized algorithm.

Project outline:

1. Background on last passage percolation: we will learn some of the results of interest in LPP on $\mathbb{Z}^2$ and its connections to broader areas of statistical physics to motivate our own study.
2. Learn some of the existing work on LPP on mean-field graphs and some classical tools from probability theory and percolation such as subadditive ergodic theorems, concentration inequalities, notions of influence, and the aforementioned connections to randomized algorithms.
3. Apply these tools to our models to obtain new and/or sharper results.

Prerequisites:

Students will be expected to have taken a probability theory class such as MATH GU4155 as well as be well-equipped in real analysis. While we will spend some time learning some tools that are required which are more specialized, students will need to be ready to learn fairly quickly.