An Introduction to Random Geometry

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Consider a graph embedded on a surface with some topology. Suppose that we are interested in its intrinsic shape and not its particular embedding, and so we view the graph modulo orientation-preserving homeomorphisms. We call this equivalence class of graphs a *planar map*. In this project, we will study *random planar maps*—namely, probability distributions on spaces of planar maps. Random planar maps are fundamental models of random discrete surfaces with applications in combinatorics, probability, physics, and computer science.

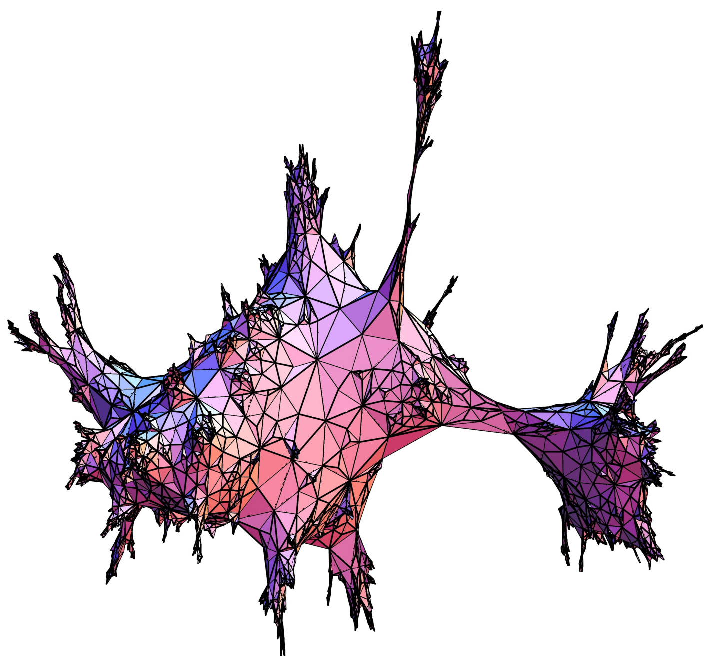

Figure 1: A representation of a random planar map with the topology of the sphere. The random planar map has been drawn so that all of its edges have approximately the same length. (Simulation by Nicolas Curien.)

In this project, we will study an example of a random planar map that we can describe via a simple algorithm. We start by fixing an integer N and a value $p \in (0,1)$, and we consider two collections of squares: the set $\mathcal{S}_{\text{test}} = \{\mathbb{S}\}$, where $\mathbb{S}$ is the unit square $[0,1]^2$, and $\mathcal{S}_{\text{terminal}} = \{\}$. At each step of the algorithm, we update these two collections of squares as follows:

- We choose $S \in \mathcal{S}_{\text{test}}$. We sample a Bernoulli random variable X_S with parameter p, independently of the random variables we previously sampled.
- We consider all the squares S' containing S (including S itself) for which we have defined $X_{S'}$.
    - If the sum of these variables is equal to N, we remove S from $\mathcal{S}_{\text{test}}$ and add it to $\mathcal{S}_{\text{terminal}}$.

– Otherwise, we divide S into four squares of equal size, and we replace S in $\mathcal{S}_{\text{test}}$ by these four squares.

An example of this algorithm and the resulting random planar map is depicted in Figure 2.

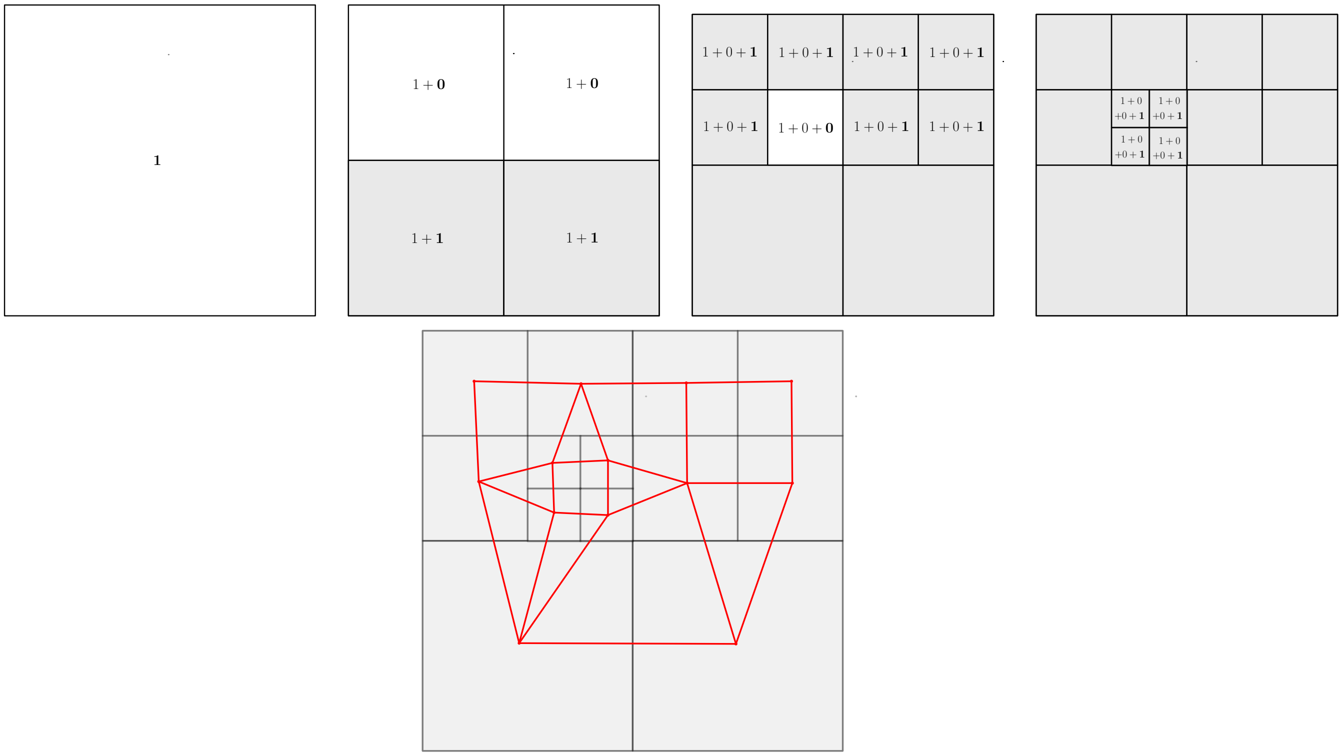

Figure 2: **Above**: an example of the square subdivision algorithm for $N = 2$. When we encounter a square S, we sample an independent Bernoulli random variable, and we sum the random variables associated to all the squares containing S. If that sum is equal to 2, the algorithm terminates at S; otherwise, we subdivide S and continue. In the above illustration, the terminal squares are shaded in gray. **Below**: the adjacency graph of terminal squares.

This example of a random planar map is a simplified version of a random planar map model that has been used to describe a type of continuum random geometry known as *Liouville quantum gravity*. In this project, we will learn some of the techniques that probabilists have used in that context and apply those tools to prove properties for the simplified model just described. In proving these results, participants in this project will be introduced to many fundamental concepts and tools in the study of random geometry, such as fractal dimension, percolation, and random walks on random planar maps. Participants are expected to have a strong background in probability and analysis.