Project: Critical fluctuations of the one-dimensional Ising model

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The *Ising model* is the classic mathematical model used to describe ferromagnetism. Although the model is old (it was introduced by Wilhelm Lenz in 1920) and easy to describe, it exhibits a very rich behavior which has not been completely understood yet.

The model is defined as follows: we have a grid in 1, 2 or 3 dimensions, in which each site is occupied by an atom whose spin takes a value either $+1$ or -1. When time evolves, each two neighboring atoms swap their spins with certain frequency, which depends on the current configuration of spins and the temperature, so that for high temperature the swaps happen more often than for low temperature (to avoid technicalities we do not provide here precise definitions). In this way we observe a time evolution of configurations of spins which randomly change. Such evolution is called the *Kawasaki dynamics*.

One of the most interesting properties of the Ising model is that it exhibits a *phase transition*. More precisely, there is a critical temperature T_c, such that the behaviors of the model for temperatures above T_c and below T_c are very different (see Figure 1, in which simulations of two-dimensional configurations for different temperatures are provided). A simple example of phase transition observed in daily life is boiling of water, i.e., when the temperature exceeds the critical value $T_c = 212°$F, water changes its state from liquid to vapor.

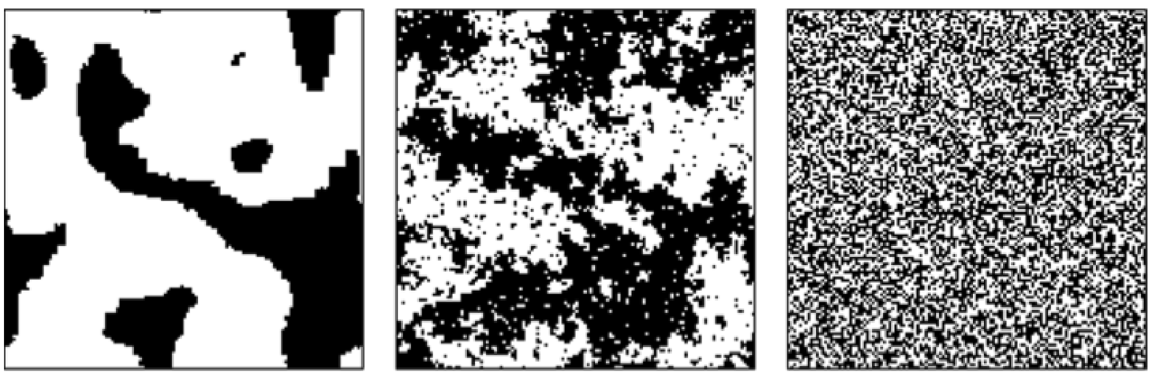

FIGURE 1. This is a simulation of two-dimensional configurations of spins for different temperatures. The spins $+1$ are drawn in black and the spins -1 are white, where we use a large scale to have a more instructive view. In the left picture the temperature is low, which means that the system is less chaotic. On the other hand, for high temperature the spins change more often, which makes the configuration on the right appear more random. The middle configuration is generated at the critical temperature.

It has been observed in several cases that the Ising model at the critical temperature can be described by certain *stochastic partial differential equations* (SPDEs) [1, 2]. However, in contrast to other types of dynamics, the Kawasaki dynamics has been studied much less, mainly due to its technical complexity. **The goal of this project is to fill in this gap and demonstrate that the one-dimensional Ising model with Kawasaki dynamics can be described at criticality by the stochastic Cahn-Hilliard equation**

$$\partial_t X = -\Delta\big(\Delta X - X^3\big) + (-\Delta)^{1/2}\xi, \tag{1}$$

driven by a random white noise ξ.[1] We consider only the one-dimensional case, because then equation (1) can be solved using the standard techniques of PDEs. In dimensions two and higher this equation is classically ill-posed and its analysis requires highly complicated techniques. In particular, analysis of a large class of such SPDEs brought a Fields Medal to Martin Hairer in 2014.

[1] Do not worry if you do not understand this equation, defining the noise ξ and solving this equation is a part of this project.

Project outline. The main steps of this project are:

(1) *Understanding the Ising model with Kawasaki dynamics and its mathematical definition.* In this step we will learn some basics of Markov processes.

(2) *Defining and solving the stochastic Cahn-Hilliard equation* (1) *in* $1D$. This step will require learning a classic method of solution of parabolic PDEs and their stochastic counterparts.

(3) *Heuristic derivation of SPDEs, describing stochastic processes such as the Kawasaki dynamics.*

(4) *Proving convergence of stochastic processes.* This will require learning the Skorokhod topology on the space of processes, the notion of weak convergence, the notion of tightness of probability measures, and some properties of martingales such as the Burkholder-Davis-Gundy inequality.

(5) *Proving that the Kawasaki dynamics converges to the stochastic Cahn-Hilliard equation* (1). In this step we combine all the learned techniques.

This project will introduce you to the topic of SPDEs, which is a very active area of research nowadays and has recently experienced a series of breakthroughs. The outline which we will pursue is a standard approach of deriving SPDEs from stochastic processes and is a backbone of many fundamental results in mathematics and physics (e.g. [1, 3]). Students are expected to know basic properties of Markov chains and Brownian motion, and to know standard methods of solution of differential equations.

References

[1] Jean-Christophe Mourrat and Hendrik Weber. Convergence of the two-dimensional dynamic Ising-Kac model to Φ^4_2. *Comm. Pure Appl. Math.*, 70(4):717–812, 2017.

[2] Giambattista Giacomin, Joel L. Lebowitz, and Errico Presutti. Deterministic and stochastic hydrodynamic equations arising from simple microscopic model systems. In *Stochastic partial differential equations: six perspectives*, volume 64 of *Math. Surveys Monogr.*, pages 107–152. Amer. Math. Soc., Providence, RI, 1999.

[3] Lorenzo Bertini and Giambattista Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.*, 183(3):571–607, 1997.