Hyperbolic volume, Mahler measure, and homology growth

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Outline

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Want: Asymptotics of $H_1(X_{G_k}^{\mathrm{br}}, \mathbb{Z})$ as $k \to \infty$.

Growth and Volume

(Kazhdan-Lück) $$\lim_{k\to\infty} \frac{b_1(X^{\mathrm{br}}_{G_k})}{[\pi : G_k]} = 0 \qquad (= L^2 - \text{Betti number}).$$

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Definition of Vol(K): $X = S^3 \setminus K$ is Haken.

$$X \setminus (\sqcup\, \text{tori}) = \sqcup\, \text{pieces}$$

each piece is either hyperbolic or Seifert-fibered.

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Theorem

$$\limsup_{k\to\infty} t(K, G_k)^{1/[\pi:G_k]} \le \exp(\mathrm{Vol}(K)).$$

Knots with 0 volumes

As a corollary, when $\mathrm{Vol}(K) = 0$, we have

$$\lim_{k\to\infty} t(K, G_k)^{1/[\pi:G_k]} = \exp(\mathrm{Vol}(K)) = 1.$$

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- $\mathrm{Vol}(K) = 0$ if and only if K is in the class
  i) containing torus knots
  ii) closed under connected sum and cabling.

More general limit: limit as $G \to \infty$

π: a countable group.
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- S': another symmetric set of generators. Then $\exists k_1, k_2 > 0$ s.t.

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- It follows that

$$\lim_{n\to\infty} \ell_S(x_n) = \infty \Longleftrightarrow \lim_{n\to\infty} \ell_{S'}(x_n) = \infty.$$

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Remark: If $\lim_{k \to \infty} \mathrm{diam} G = \infty$ then $\cap G_k = \{1\}$ (co-final).

Homology Growth and Volume

Conjecture

("volume conjecture") For every knot $K \subset S^3$,

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- To prove the conjecture one needs to find $\{G_k\}$ – finite index normal subgroups of π s. t. $\lim_k \mathrm{diam}(G_k) = \infty$ and

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It is unlikely that for any sequence G_k of normal subgroups s.t. $\lim \mathrm{diam} G_k = \infty$ one has (*). Which $\{G_k\}$ should we choose?

Expander family

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(*) *holds for the Long-Lubotzky-Reid sequence* $\{G_k\}$.

Justification: to follow.

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$$0 \to C_n \stackrel{\partial_n}{\to} C_{n-1} \stackrel{\partial_{n-1}}{\to} \dots C_1 \stackrel{\partial_1}{\to} C_0 \to 0.$$

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$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0.$$

$$\tau(\mathcal{C}) = \left[\frac{\partial_2(c_2)\partial^{-1}c_0}{c_1} \right]$$

Here $[a/b]$ is the determinant of the change matrix from b to a.

Torsion of chain of Hilbert spaces

$\mathcal{C}$: complex of finite dimensional Hilbert spaces over $\mathbb{C}$; acyclic.
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More specifically,

$$C_i = \mathbb{Z}[\pi]^{n_i}, \quad \text{free } \mathbb{Z}[\pi]-\text{module, or } C_i = \ell^2(\pi)^{n_i}$$

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- Need to define what is the determinant of a matrix $A \in \text{Mat}(m \times n, \mathbb{Z}[\pi])$.

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- Adjoint operator: $x = \sum c_g g \in \mathbb{C}[\pi]$, then $x^* = \sum \bar{c}_g g^{-1}$.
- Similarly to the finite group case, define $\forall g \in \pi$,

$$\operatorname{tr}(g) = \delta_{g,1}$$

$$\forall x \in \mathbb{C}[\pi], \operatorname{tr}(x) = \langle x, 1\rangle = \text{coeff. of 1 in } x.$$

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- (not rigorous) Define $\det(A)$ using

$$\begin{aligned} \log \det A &= \mathrm{tr} \log A \\ &= -\mathrm{tr} \sum_{p=1}^{\infty} (I - A)^p / p \\ &= -\sum \frac{\mathrm{tr}[(I - A)^p]}{p}. \end{aligned}$$

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- Convergence of the RHS?

Fuglede-Kadison-Lück determinant for $A \in \mathrm{Mat}(m \times n, \mathbb{C}[\pi])$

- $B := A^*A$, where $(A^*)_{ij} := (A_{ji})^*$. $\ker(B) = \ker A$, $B \geq 0$.

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Most interesting case: A is injective ($b = 0$), $m = n$, but not invertible.

FKL determinant – Example: Finite group

- $D \in \mathrm{Mat}(n \times n, \mathbb{C})$. Let $p(\lambda) = \det(\lambda I + D)$.

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- $|\pi| < \infty, \quad A \in \mathrm{Mat}(m \times n, \mathbb{C}[\pi])$. Then A is given by a matrix $D \in \mathrm{Mat}(m|\pi| \times n|\pi|, \mathbb{C})$.

$$\det_\pi A = \left(\det{}'(D^* D)\right)^{1/2|\pi|}.$$

FKL determinant– Example: $\pi = \mathbb{Z}^\mu$

- $f(t_1^{\pm 1}, \dots, t_\mu^{\pm 1}) \in \mathbb{C}[\mathbb{Z}^\mu] \equiv \mathbb{C}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$.
  Assume $f \neq 0$. f: 1×1 matrix.

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  Assume $f \neq 0$. f: 1×1 matrix.

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- $f(t) \in \mathbb{Z}[t^{\pm 1}]$, $f(t) = a_0 \prod_{j=1}^n (t - z_j), z_j \in \mathbb{C}$. Then

$$M(f) = a_0 \prod_{|z_j|>1} |z_j|.$$

Outline

L^2-Torsion, L^2-homology of $\mathbb{C}[\pi]$- complex

$$\mathcal{C}: \quad 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots C_1 \xrightarrow{\partial_1} C_0 \to 0.$$

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- If $C(\tilde{X})$ is acyclic and of det-class for one triangulation, then it is acyclic and of det-class for any other triangulation, and $\tau^{(2)}(\tilde{X})$ of the two triangulations are the same: we can define $\tau^{(2)}(\tilde{X})$.

L^2-Torsion of knots: universal covering

- K a knot in S^3. $X = S^3 - K$, $\tilde{X}$: universal covering. $\pi = \pi_1(X)$. Then $\tilde{X}$ is a π-space with quotient X.

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- Theorem (Lück-Schick)

$$\log \tau^{(2)}(K) = -\mathrm{Vol}(K).$$

based on results of Burghelea-Friedlander-Kappeler-McDonald, Lott, and Mathai.

L^2-Torsion of knots: computing using knot group

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  Y has 1 0-cell, $(n+1)$ 1-cells, and n 2-cells. $\tilde{Y}$: universal covering.

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$$\partial_1 = \begin{pmatrix} a_1 - 1 \\ a_2 - 1 \\ \vdots \\ a_{n+1} - 1 \end{pmatrix}, \quad \partial_2 = \left(\frac{\partial r_i}{\partial a_j} \right) \in \mathrm{Mat}(n \times (n+1), \mathbb{Z}[\pi])$$

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By definition

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Theorem

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L^2-Torsion for abelian covering of links

L a link of μ components. $X = S^3 \setminus L$.

$$\pi = \pi_1(X).$$

Abelianization map $\mathrm{ab} : \pi \to \mathbb{Z}^\mu$.
$\tilde{X}^{\mathrm{ab}}$: abelian covering corresponding to $\ker(\mathrm{ab})$, $\mathbb{Z}^\mu$-space.
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Proposition

$C(\tilde{X}^{\mathrm{ab}})$ is of det-class. $C(\tilde{X}^{\mathrm{ab}})$ is acyclic if and only if $\Delta_0(L) \neq 0$. If $\Delta_0(L) \neq 0$

$$\tau^{(2)}(\tilde{X}^{\mathrm{ab}}) = \frac{1}{M(\Delta_0(L))}.$$

If $\mu = 1$, then $\Delta_0 \neq 0$ always.

Outline

Finite quotient

$\mathcal{C}$: $\mathbb{Z}[\pi]$-complex, free finite rank. G a normal subgroup, $\pi \to \Gamma = \pi/G$.

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Question When

$$\lim_{\mathrm{diam} G \to \infty} t(\mathcal{C}, G)^{1/[\pi:G]} = \tau^{(2)} \mathcal{C}?$$

Full result for $\pi = \mathbb{Z}$

Theorem

$\pi = \mathbb{Z}$. $\quad G_k = k\mathbb{Z} \subset \mathbb{Z}$.

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- Proof of theorem used a special case, a result of Lück (Riley, Gonzalez-Acuna, and Short) based on Gelfond-Baker theory of diophantine approximation): $f \in \mathbb{Q}[\mathbb{Z}]$, then

$$\det{}_{\mathbb{Z}} f = \lim_{n\to\infty} \det{}_{\mathbb{Z}/k}(f_{\mathbb{Z}/k})$$

and a result relating $\det_{\mathbb{Z}_k}$ to $|\mathrm{Tor}|$.

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Consider only lattice $G < \mathbb{Z}^\mu$ such that $\operatorname{rk} G = \mu$.

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$A \in \operatorname{Mat}(m \times n, \mathbb{C}[\mathbb{Z}^\mu])$. *Then*

$$\det_{\mathbb{Z}^\mu} A = \limsup_{\operatorname{diam} G \to \infty} \det_{\mathbb{Z}^\mu / G}(A_G).$$

Application: Link case

L: μ-component link in S^3. Assume $\Delta_0(L) \neq 0$ (always the case if $\mu = 1$).
G a lattice in $\mathbb{Z}^\mu$ of rank μ. X_G^{br}: branched G-covering of $X = S^3 \setminus L$.

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Corollary

(Silver-Williams)

$$M(\Delta_0(L)) = \limsup_{\mathrm{diam} G \to \infty} t(L, G)^{1/[\mathbb{Z}^\mu : G]}.$$

If $\mu = 1$*, then* lim sup *can be replaced by* lim*.*

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- For knots: Question of Gordon, answered by Riley and by Gonzalez-Acuna and Short.

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Proposition

$$\limsup_{\text{diam}G\to\infty} t(L,G)^{1/[\mathbb{Z}^\mu:G]} \geq M(\Delta(L)).$$

Used a theorem of Schinzel-Bombieri-Zannier (2000) on co-primeness of specializations of multivariable polynomials.

Knot case: Expander family

$$0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0, \qquad \tau^{(2)} = \frac{\det_\pi \partial_1}{\det_\pi \partial_2}$$

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$$\exp(\mathrm{Vol}(K)) = \limsup_{\mathrm{diam}G \to \infty} t(K, G)^{1/[\pi:G]}$$

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A convergence criterion of Lück: For $A \in \mathrm{Mat}(m \times n, \mathbb{Z}[\pi])$, $B = A^* A$, if the eigenvalues of the B_G near 0 “behaves well”, then

$$\det_\pi A = \lim_{G \to \infty} \det_{\pi/G} A_G.$$

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A convergence criterion of Lück: For $A \in \mathrm{Mat}(m \times n, \mathbb{Z}[\pi])$, $B = A^* A$, if the eigenvalues of the B_G near 0 "behaves well", then

$$\det_\pi A = \lim_{G \to \infty} \det_{\pi/G} A_G .$$

For expander family, requirements of Lück criterion are satisfied *trivially* for $A = \partial_1$:
∂_1 can be approximated by finite quotients (from expander family).

Knot case: Expander family

$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0, \qquad \tau^{(2)} = \frac{\det_\pi \partial_1}{\det_\pi \partial_2}$$

One can prove the volume conjecture

$$\exp(\mathrm{Vol}(K)) = \limsup_{\mathrm{diam} G \to \infty} t(K, G)^{1/[\pi:G]}$$

if one can approximate both $\det_\pi \partial_1, \det_\pi \partial_2$ by finite quotients.

A convergence criterion of Lück: For $A \in \mathrm{Mat}(m \times n, \mathbb{Z}[\pi])$, $B = A^* A$, if the eigenvalues of the B_G near 0 “behaves well”, then

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For expander family, requirements of Lück criterion are satisfied *trivially* for $A = \partial_1$:

∂_1 can be approximated by finite quotients (from expander family).

Same for ∂_2? Yes $\Longrightarrow$ ‘volume conjecture” for hyperbolic knots.

THANK YOU!