Diffusion scaling of a limit-order book model

Steven E. Shreve
Department of Mathematical Sciences
Carnegie Mellon University
shreve@andrew.cmu.edu

nearly complete work with
Christopher Almost
John Lehoczky
Xiaofeng Yu

Thera Stochastics
In Honor of Ioannis Karatzas
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Goals of this work

Build a "zero-intelligence" Poisson model of the limit-order book and determine its diffusion limit.

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- Diffusion limit — Evolution of the limiting limit-order book is described in terms of Brownian motions.
- Computation of statistics – Use the limiting limit-order model to compute statistics of model dynamics. □

Partial history

- CONT, R., STOIKOV, S. & TALREJA, R. (2010) "A stochastic model for order book dynamics," *Operations Research* **58**, 549–563.
  Poisson arrivals of buy and sell orders and exponential waiting times before cancellations. Use Laplace transforms to compute statistics of the order book dynamics.

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- CONT, R. & DE LARRARD, A. (2013) "Price dynamics in a Markovian limit order market," *SIAM J. Financial Mathematics* **4**, 1–25.
  Always a one tick spread and orders queue only at the best bid and best ask prices. If one of these is depleted, both move one tick and the book reinitializes. Derive the diffusion-scaled limit.

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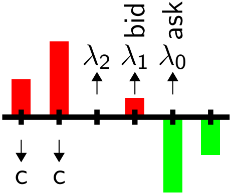
- The pre-limit model, which the limiting model approximates, has more realistic behavior. For the parameters considered here, there is a one-tick spread 76% of the time.
- We present some computations in the limiting model. □

Arrivals and cancellations of buy orders



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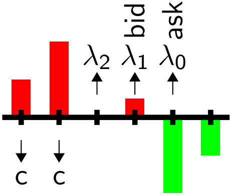

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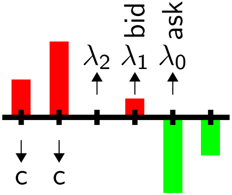

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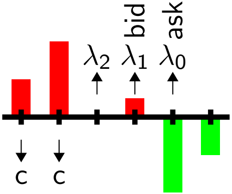

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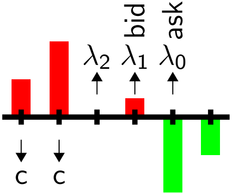

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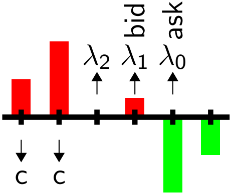

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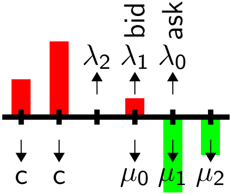
- All processes are independent of one another. □

Arrivals and cancellations of sell orders



- All orders are of size 1.
- Poisson arrivals of market sells at rate μ_0. These execute at the best bid price.
- Poisson arrivals of limit sells at one and two ticks above the best bid price at rates μ_1 and μ_2, respectively.
- Cancellations of limit sells two or more ticks above the best ask price, at rate $\theta/\sqrt{n}$ per order.
- All processes are independent of one another. □

Constraints on parameters

In order to have a diffusion limit, among the six parameters λ_0, λ_1, λ_2, μ_0, μ_1, and μ_2, there are three degrees of freedom. Let a and b be positive constants satisfying $a + b > ab$. Then

$$\begin{aligned}
\lambda_1 &= (a-1)\lambda_0, \\
\lambda_2 &= (a+b-ab)\lambda_0, \\
\mu_1 &= (b-1)\mu_0, \\
\mu_2 &= (a+b-ab)\mu_0, \\
a\lambda_0 &= b\mu_0.
\end{aligned}$$

In addition to a and b, there is a scale parameter, which can be set by choosing μ_0.

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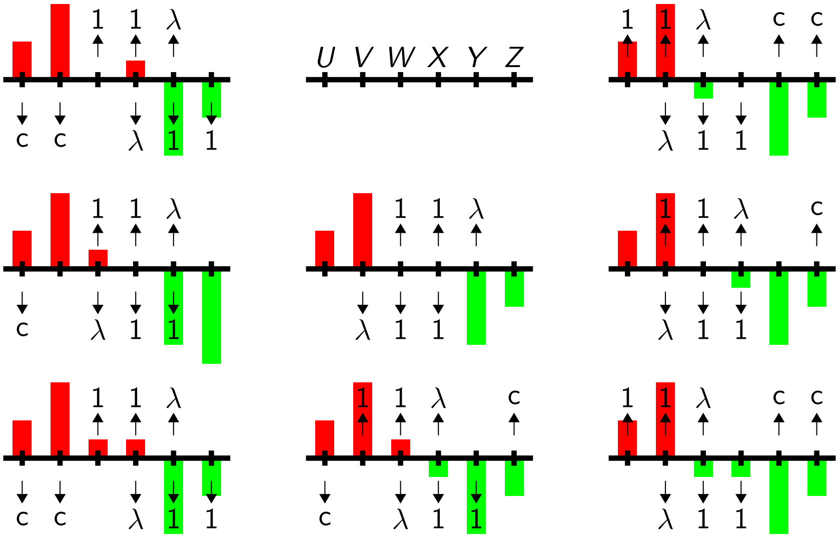
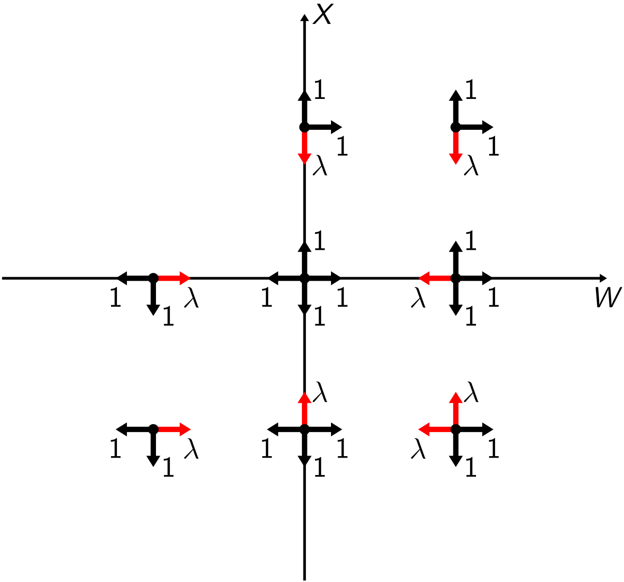
To simplify the presentation, we set

$$\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1,$$
$$\lambda_0 = \mu_0 = \lambda := (1+\sqrt{5})/2.$$

Limit-order book arrivals and departures

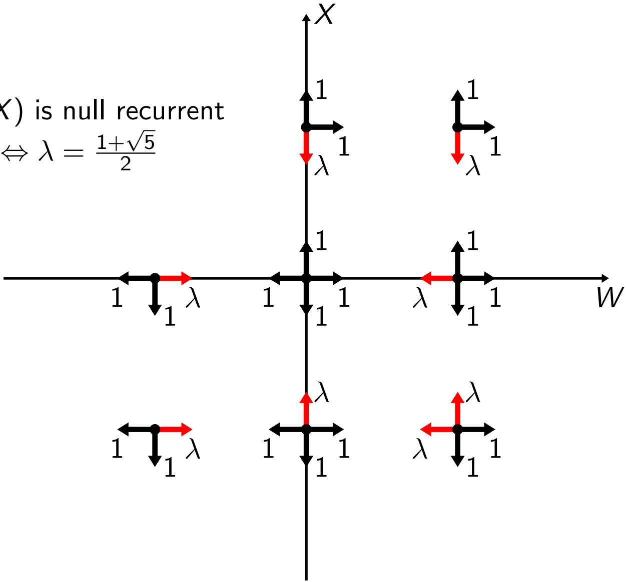


Transitions of (W, X)



Transitions of (W, X)

(W, X) is null recurrent
$\Leftrightarrow \lambda = \frac{1+\sqrt{5}}{2}$



Split Brownian motion

The diffusion scaling of a generic process Q is defined to be

$$\widehat{Q}^n(t) := \frac{1}{\sqrt{n}} Q(nt).$$

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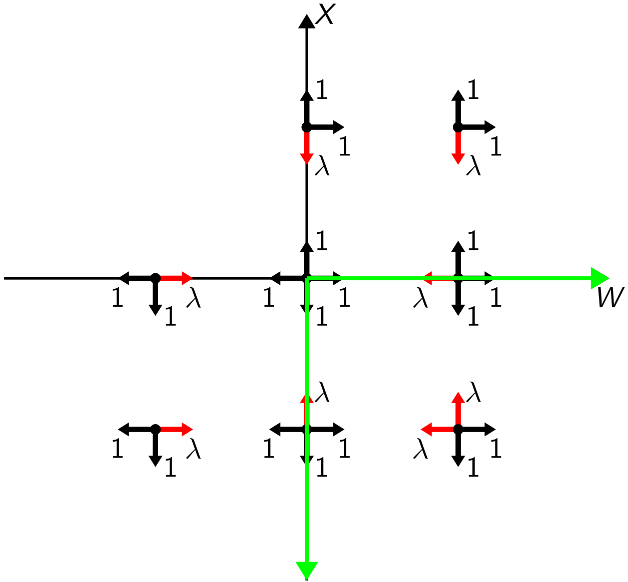
Theorem

Conditional on the bracketing processes V and Y remaining nonzero, $(\widehat{W}^n, \widehat{X}^n)$ converges in distribution to a split Brownian motion

$$(W^*, X^*) = (\max\{G^*, 0\}, \min\{G^*, 0\}),$$

where G^* is a one-dimensional Brownian motion with variance 4λ per unit time. □

Split Brownian motion



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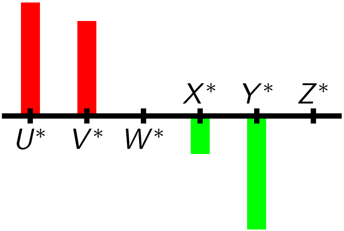

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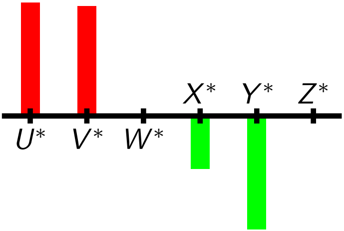

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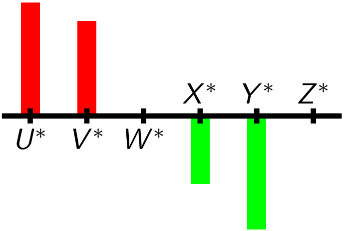

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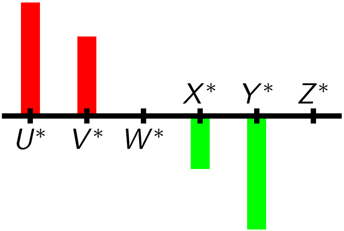

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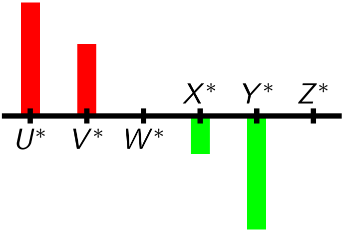

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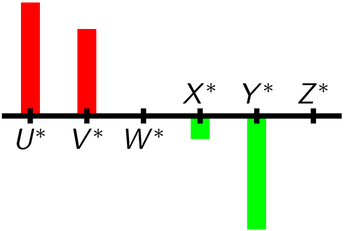

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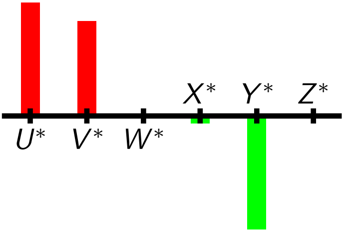

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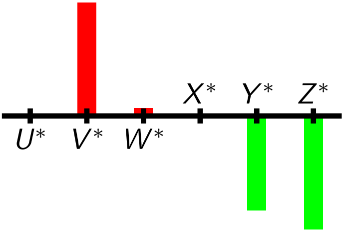

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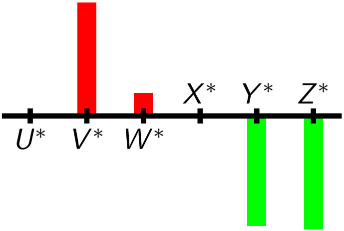

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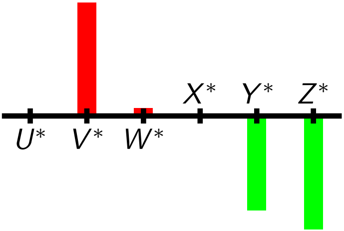

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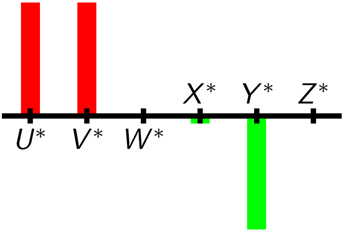

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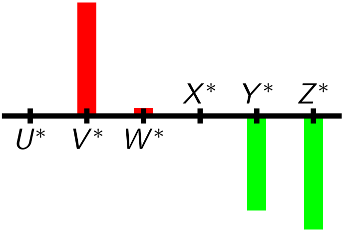

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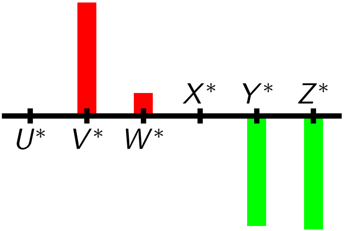

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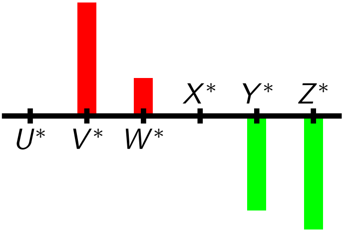

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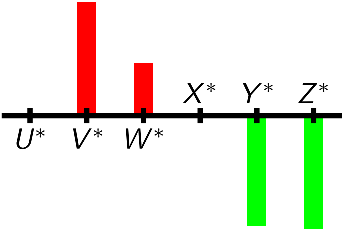

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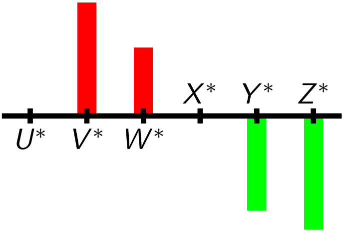
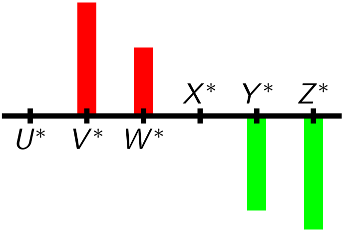


The other queues



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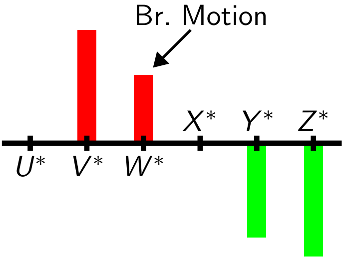
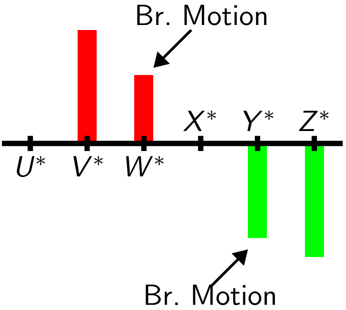


$\frac{d}{dt}\langle W^*, W^*\rangle_t = 4\lambda,$

The other queues



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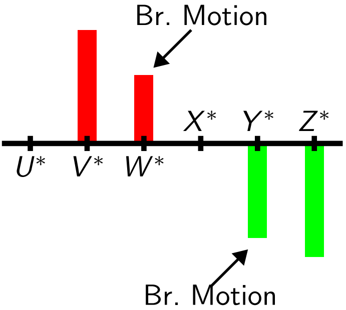

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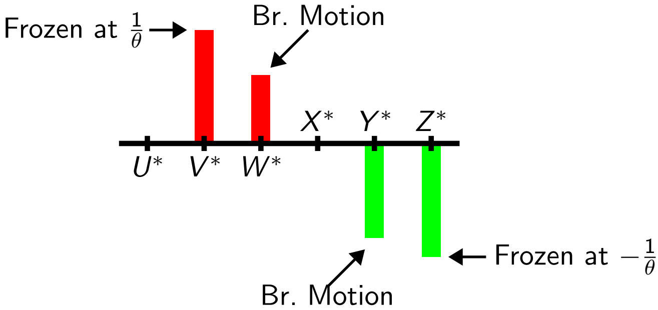

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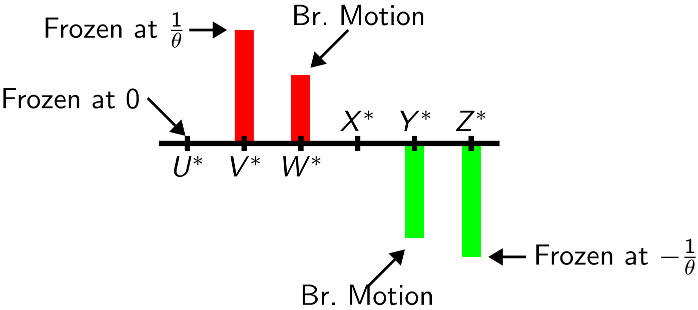

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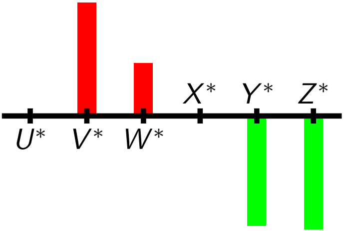

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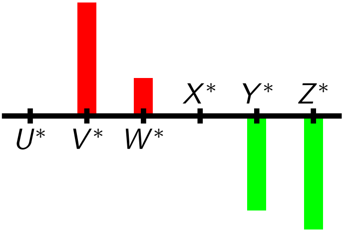

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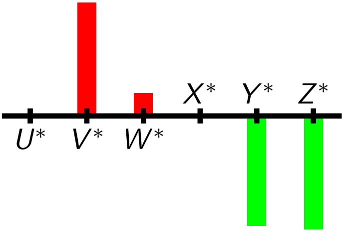

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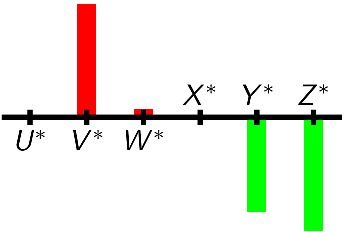

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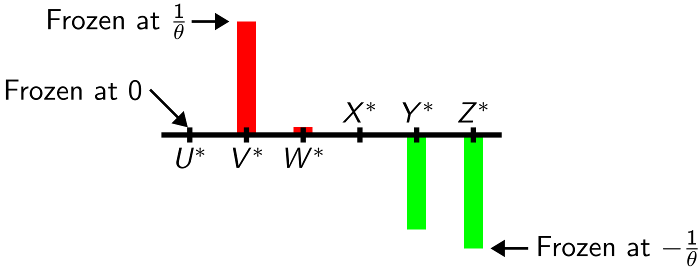

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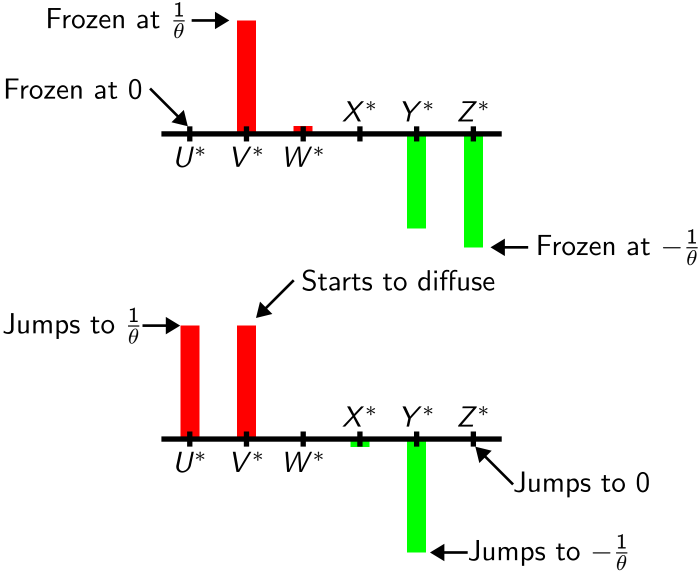

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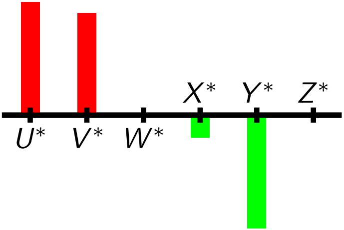
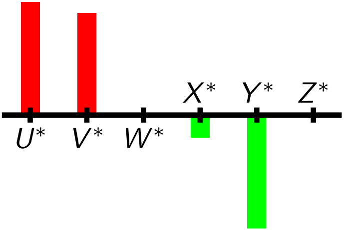


V^* and X^* are in a race to zero.

The other queues

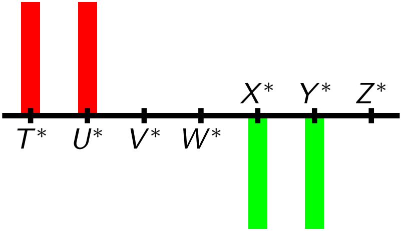


V^* and X^* are in a race to zero.

A. Metzler, *Stat. & Probab. Letters*, 2010: "On the first passage problem for correlated Brownian motion." □

The other queues

Suppose V^* wins.



- Reset the "bracketing processes" to be U^* and X^*.
- (V^*, W^*) begins executing a split Brownian motion. □

Snapped Brownian motion

Let's consider the V^* process in more detail.

Snapped Brownian motion

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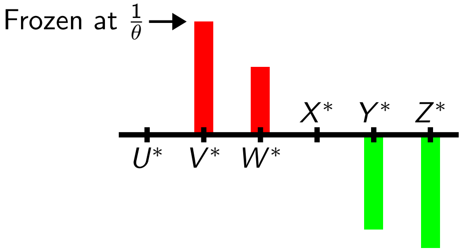
As long as the "bracketing processes" V^* and Y^* remain nonzero, (W^*, X^*) executes a split Brownian motion:

$$(W^*, X^*) = (\max\{G^*, 0\}, \min\{G^*, 0\}),$$

where G^* is a one-dimensional Brownian motion with variance 4λ per unit time.

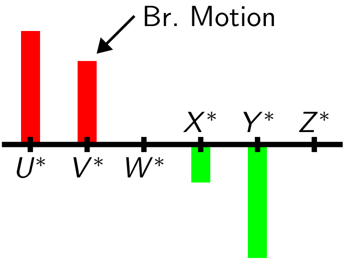


Snapped Brownian motion

Still have the split Brownian motion,

$$(W^*, X^*) = (\max\{G^*, 0\}, \min\{G^*, 0\}),$$

but now V^* is diffusing.



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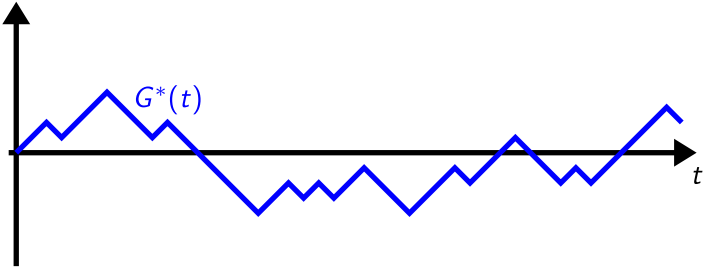
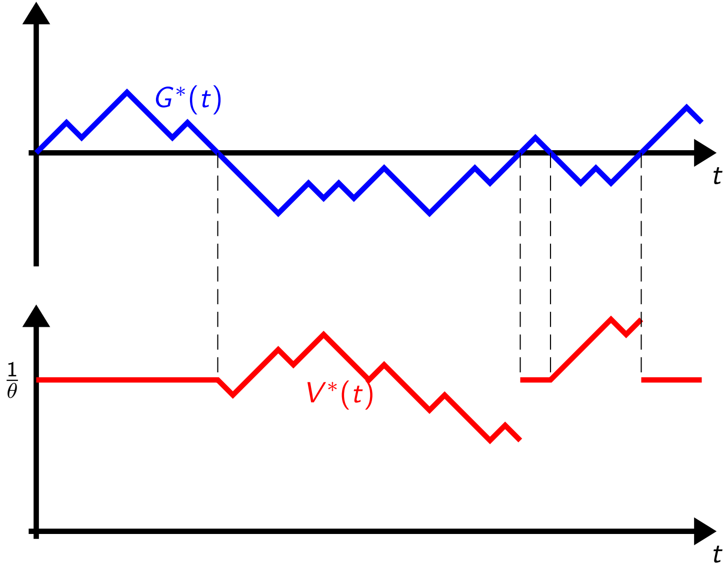


Snapped Brownian motion



Summary of properties of the limiting model

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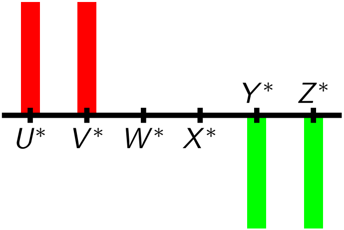
- When the queue at the best bid or the best ask is depleted, we have a three-tick spread.
- We transition through the three-tick spread using the concept of a snapped Brownian motion. □

Renewal states



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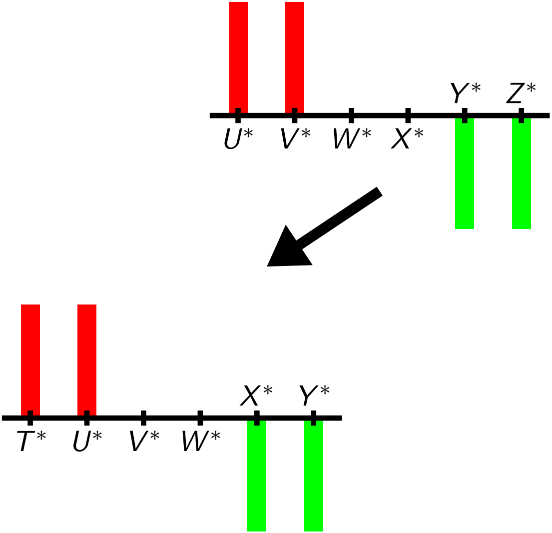

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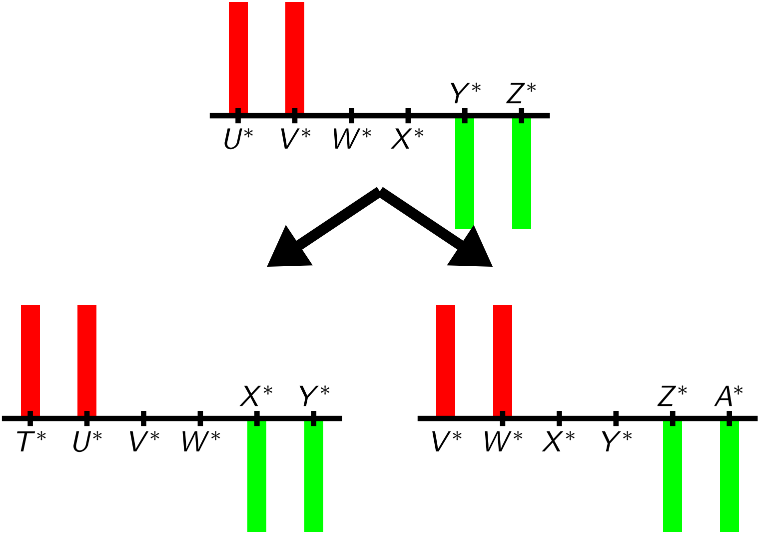

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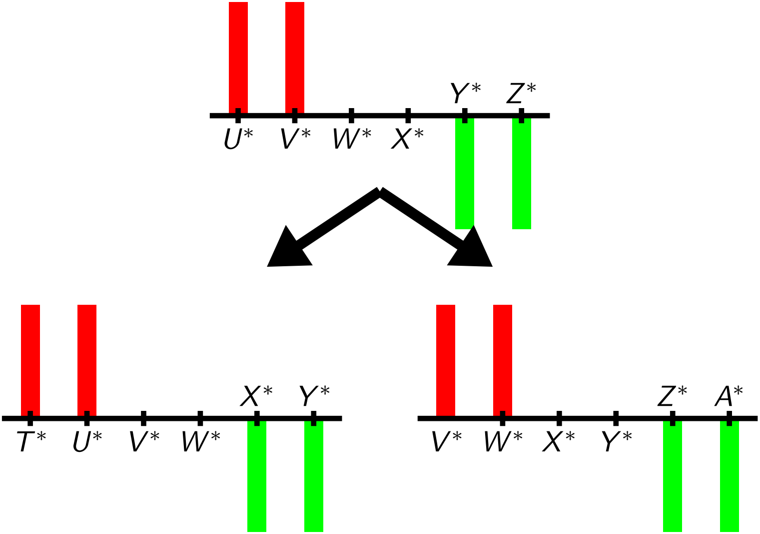

Which way? How long?

How long to transition between renewal states?

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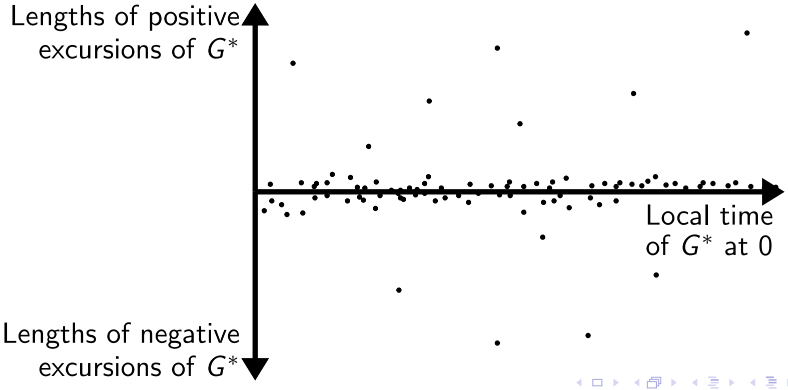

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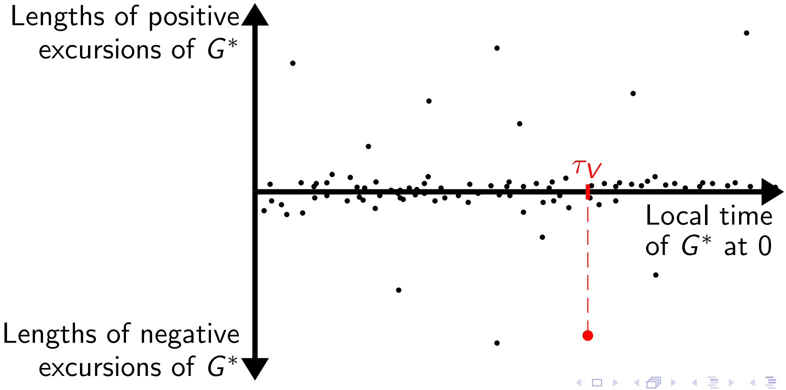

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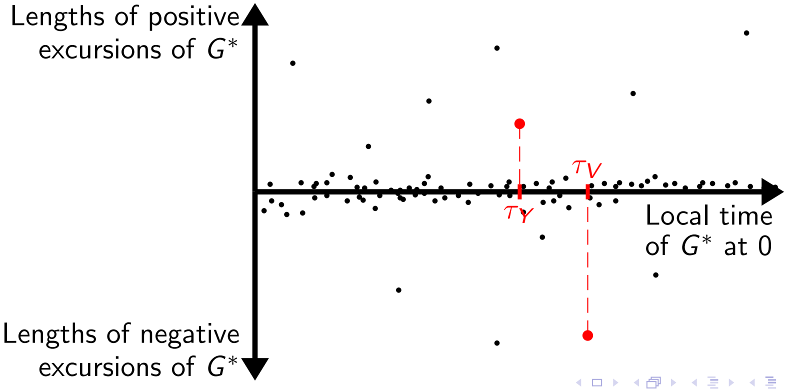

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- For (ii), we observe that the distribution of the length of the "last excursion" is $\mu_0(d\ell)/\mu_0((0,\infty))$.
- Adapt Metzler again to compute the distribution of the elapsed time T_2 in the "last excursion," conditioned on its length. □

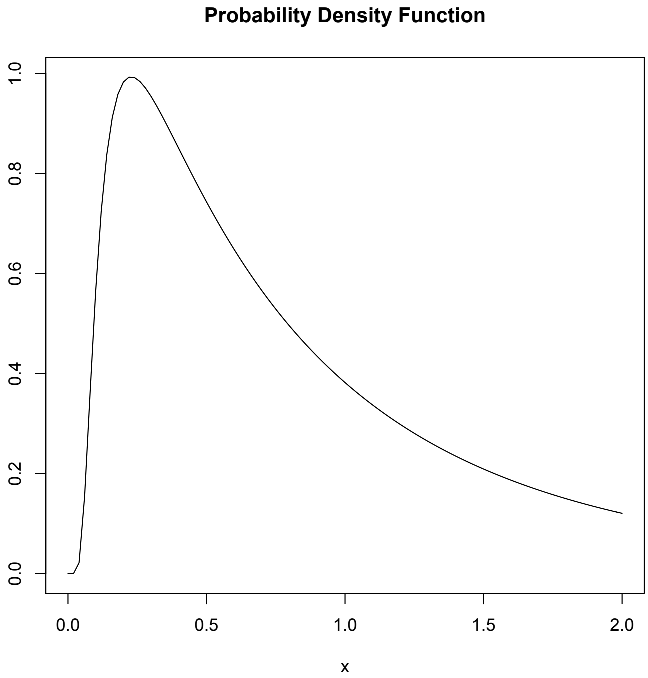
Calculation of renewal time distribution

The moment-generating function of $T_1 + T_2$ is

$$\mathbb{E}\left[e^{-\alpha(T_1+T_2)}\right]$$

$$= \int_0^\infty \int_0^\ell e^{-\alpha s} \frac{p(s,\ell)}{\sqrt{2\pi\ell^3}}\, ds\, d\ell \Bigg/ \left(\sqrt{\frac{\alpha}{2}} + \int_0^\infty e^{-\alpha\ell} \frac{p(\ell)\, d\ell}{2\sqrt{2\pi\ell^3}}\right),$$

where $p(s,\ell)$ is the conditional density in s of the elapsed time T_2 given that the “last excursion” has length ℓ.



Probability Density Function
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
x

Concluding remarks

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MARTINGALES AND STOCHASTIC INTEGRALS IN THE THEORY OF CONTINUOUS TRADING

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Graduate School of Business, Stanford University, Stanford, CA 94305, U.S.A.

Stanley R. PLISKA
Northwestern University, Evanston, IL 20601, U.S.A.

Stochastic Processes and their Applications 15 (1983) 313–316
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SHORT COMMUNICATION

A STOCHASTIC CALCULUS MODEL OF CONTINUOUS TRADING: COMPLETE MARKETS

J. Michael HARRISON
Graduate School of Business, Stanford University, Stanford, CA 94305, U.S.A.

Stanley R. PLISKA
Department of Industrial Engineering and Management Science, Northwestern University, Evanston, IL 60201, U.S.A.

1.6. Outline of the paper

This paper is aimed at readers with a good command of probability and stochastic processes, but no particular knowledge of economics. On the former dimension, we assume familiarity with the Strasbourg theory of martingales and stochastic integration, as developed in the definitive treatise by Meyer [32]. This assumption is perhaps unrealistic, but we cannot provide a systematic tutorial on stochastic integrals and an adequate treatment of our nominal subject matter in a reasonable amount of space. (Also, the former task is best left to others. We are working dangerously close to the boundaries of our knowledge as things stand.) Most of this paper will be accessible to those who know about stochastic integrals with respect to Brownian motion, and the rest should come into focus after a little study of the relevant foundational material. (On first reading, specialize general results to the case where S is an Ito process.) To facilitate such study, we consistently refer to Meyer [32] by page number for basic definitions and standard results, and his notation and terminology are used wherever possible. For a nice overview of the Strasbourg approach to stochastic integration, plus some new results and illuminating commentary, see the recent survey by Dellacherie [9] in this journal. A comprehensive treatment of stochastic calculus is given by Jacod [18], and it appears that the second volume of Williams [38] will be another good sourcebook on martingales and stochastic integrals in the Strasbourg

Φρέδος Παπαγγέλου

Fredos Papangelou
Emeritus Professor
School of Mathematics
University of Manchester

Vorlesungen über Maβtheorie von F. Papangelou

"Sei I_n^* das nach beiden Seiten um seine zweifache Länge erweiterte Intervall I_n."

Vorlesungen über Maβtheorie von F. Papangelou

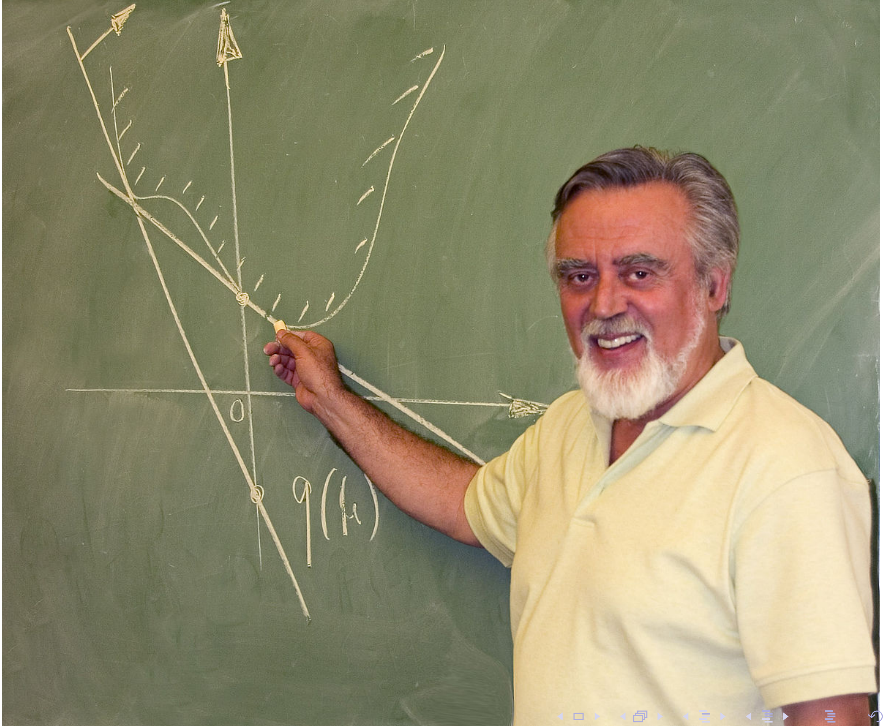
> *"Sei* I_n^* *das nach beiden Seiten um seine zweifache Länge erweiterte Intervall* I_n*."*

English translation:

> *"Let* I_n^* *be the toward-both-sides-for-its-doubled-length-extended interval* I_n*."*

Δημήτρης Μπερτσεκάς

Dimitri Bertsekas
Jerry Mcafee Professor
Electrical and Computer Engineering
Massachusetts Institute of Technology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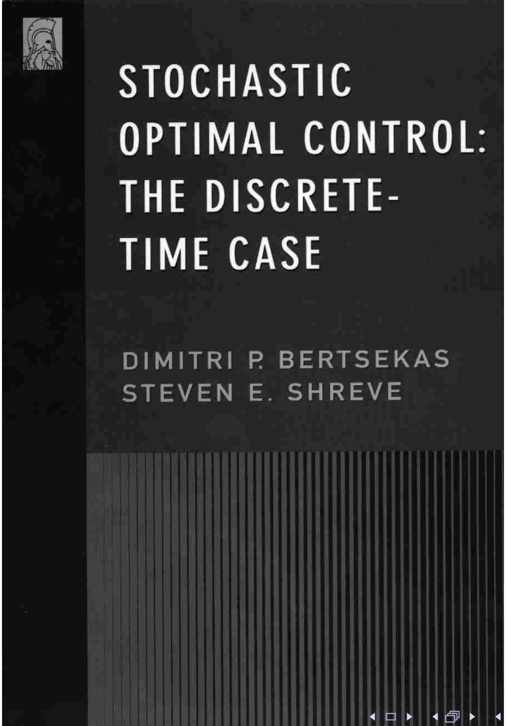
STOCHASTIC
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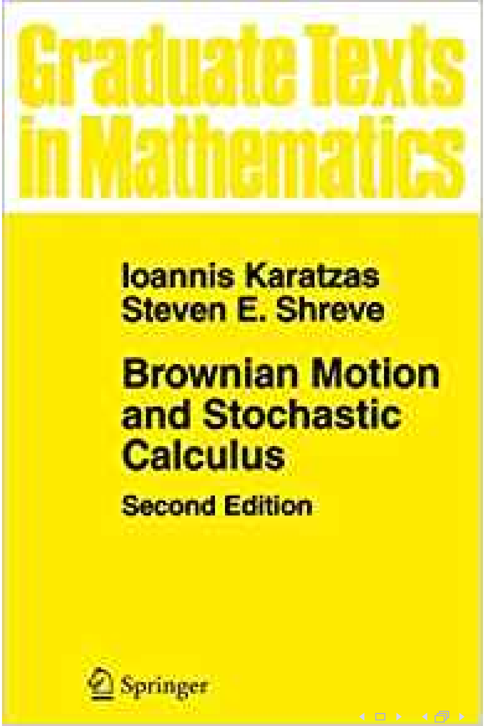
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"Martingales sprang fully armed from the forehead of Joseph Doob." — Karatzas

If you enounter Greeks bearing gifts....

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welcome them with open arms.