Asset pricing under optimal contracts

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joint work with

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Motivation and overview

- Existing literature:
  either
  - Prices are fixed, optimal contract is found
  or
  - Contract is fixed, prices are found in equilibrium
- An exception: Buffa-Vayanos-Woolley 2014 [BVW 14]
- However, [BVW 14] still severely restrict the set of admissible contracts
- We allow more general contracts and explore equilibrium implications

Literature

- Fixed contracts:

  Brennan (1993)
  Cuoco-Kaniel (2011)
  He-Krishnamurthy (2011)
  Lioui and Poncet (2013)
  Basak-Pavlova (2013)

- Fixed prices:

  Sung (1995)
  Ou-Yang (2003)
  Cadenillas, Cvitanić and Zapatero (2007)
  Leung (2014)
  Cvitanić, Possamai and Touzi, CPT (2016, 2017)

Buffa-Vayanos-Woolley 2014 [BVW 14]

- Optimal contract is obtained within the class

$$\text{compensation rate} = \phi \times \text{ portfolio return} - \chi \times \text{ index return.}$$

Our questions:

1. What is the optimal contract when investors are allowed to optimize in a larger class of contracts?
   (Linear contract is optimal in [Holmstrom-Milgrom 1987])

2. What are the equilibrium properties?

As shown in CPT (2016, 2017) ...

- The optimal contract depends on the output, its **quadratic variation**, the contractible sources of risk (if any), and the **cross-variations** between the output and the risk sources.

Our results

- Computing the optimal contract and equilibrium prices
- Optimal contract rewards Agent for taking specific risks and not only the systematic risk
- Stocks in large supply have high risk premia, while stocks in low supply have low risk premia
- Equilibrium asset prices distorted to a lesser extent:

  Second order sensitivity to agency frictions compared to the first order sensitivity in [BVW 14].

Outline

Assets

Riskless asset has an exogenous constant risk-free rate r.

Prices of N risky assets will be determined in equilibrium.

Dividend of asset i is given by

$$D_{it} = a_i p_t + e_{it},$$

where p and e_i follow Ornstein-Uhlenbeck processes

$$dp_t = \kappa^p(\bar{p} - p_t)dt + \sigma_p dB_t^p,$$
$$de_{it} = \kappa_i^e(\bar{e}_i - e_{it})dt + \sigma_{ei} dB_{it}^e.$$

Vector of asset excess returns per share

$$dR_t = D_t dt + dS_t - rS_t dt.$$

The excess return of index

$$I_t = \eta' R_t,$$

where $\eta = (\eta_1, \ldots, \eta_N)'$ are the numbers of shares of assets in the market.

Available shares

Number of shares available to trade:
$\theta = (\theta_1, \ldots, \theta_N)'$
(Some assets may be held by buy-and-hold investors.)

We assume that η and θ are not linearly dependent. (Manager provides value to Investor.)

Portfolio manager

Portfolio manager's wealth process follows

$$d\bar{W}_t = r\bar{W}_t dt + (b\, m_t - \bar{c}_t)dt + dF_t,$$

- $\bar{c}_t$ is Manager's consumption rate
- F_t is the cumulative compensation paid by Investor
- $b\, m_t$ is the private benefit from his shirking action m_t, $b \in [0, 1]$, [DeMarzo-Sannikov 2006]
- No private investment
- Chooses portfolio Y for Investor

Investor

The reported portfolio value process:

$$G = \int_0^{\cdot} (Y_s' dR_s - m_s ds).$$

Investor observes only G and I

Her wealth process follows

$$dW_t = rW_t dt + dG_t + y_t dI_t - c_t dt - dF_t,$$

- Y_t is the vector of the numbers of shares chosen by Manager
- y_t is the number of shares of index chosen by Investor
- c_t is Investor's consumption rate
- m_t is Manager's shirking action, assumed to be nonnegative

Manager's optimization problem

Manager maximizes utility over intertemporal consumption:

$$\bar{V} = \max_{\bar{c},m,Y} \mathbb{E}\Big[\int_0^\infty e^{-\bar{\delta}t} u_A(\bar{c}_t)dt\Big],$$

- $\bar{\delta}$ is Manager's discounting rate
- $u_A(\bar{c}) = -\frac{1}{\bar{\rho}} e^{-\bar{\rho}\bar{c}}$

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If Manager is not employed by Investor, he maximizes

$$\bar{V}^u = \max_{\bar{c}^u, Y^u} \mathbb{E}\Big[\int_0^{\infty} e^{-\bar{\delta}t} u_A(\bar{c}_t^u)dt\Big]$$

subject to budget constraint

$$d\bar{W}_t = r\bar{W}_t + Y_t^u dR_t - \bar{c}_t^u dt.$$

Manager takes the contact if $\bar{V} \geq \bar{V}^u$.

Investor's maximization problem

Investor maximizes utility over intertemporal consumption:

$$V = \max_{c,F,y} \mathbb{E}\Big[\int_0^\infty e^{-\delta t} u_P(c_t) dt \Big],$$

- δ is Investor's discounting rate
- $u_P(c) = -\frac{1}{\rho} e^{-\rho c}$

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If Investor does not hire Manager, she maximizes

$$V^u = \max_{c^u,y^u} \mathbb{E}\Big[\int_0^\infty e^{-\delta t} u_P(c_t^u)dt\Big]$$

subject to budget constraint

$$dW_t = rW_t + y_t^u dI_t - c_t^u dt.$$

Investor hires Manager if $V \geq V^u$.

Equilibrium

A price process S, a contract F in a class of contracts $\mathbb{F}$, and an index investment y, form an equilibrium if

1. Given S, $(F, \mathbb{F})$, and y, Manager takes the contract, and $Y = \theta - y\,\eta$ solves Manager's optimization problem.

2. Given S, Investor hires Manager, and (F, y) solves Investor's optimization problem, and F is the optimal contract in $\mathbb{F}$.

Outline

Asset prices

There exists an equilibrium with asset prices $S_{it} = a_{0i} + a_{pi} p_t + a_{ei} e_{it}$ (assuming θ and η are not linearly dependent.)

Setting $a_p = (a_{p1}, \ldots, a_{pN})'$ and $a_e = diag\{a_{e1}, \ldots, a_{eN}\}$, we have

$$a_{pi} = \frac{a_i}{r + \kappa^p} \quad a_{ei} = \frac{1}{r + \kappa_i^e}, \quad i = 1, \ldots, N,$$

(assuming the matrix $\Sigma_R = a_p \sigma_p^2 a_p' + a_e' \sigma_E^2 a_e$ is invertible.)

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Notation:

$$Var^\eta = \eta' \Sigma_R \eta, \quad Covar^{\theta,\eta} = \eta' \Sigma_R \theta,$$

CAPM beta of the fund portfolio: $\beta^\theta = \dfrac{Covar^{\theta,\eta}}{Var^\eta}.$

Asset Returns

Asset excess returns are

$$\mu - r = r\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\Sigma_R\theta + r\mathscr{D}_b\Sigma_R(\theta - \beta^\theta\eta),$$

where

$$\mathscr{D}_b = (\rho+\bar{\rho})\Big(b - \frac{\rho}{\rho+\bar{\rho}}\Big)_+^2.$$

- When $b \in [0, \frac{\rho}{\rho+\bar{\rho}}]$, the first best is obtained.
- When $\frac{\theta_i}{\eta_i} > \beta^\theta$, risk premium of asset i increases with b.

  When $\frac{\theta_i}{\eta_i} < \beta^\theta$, risk premium of asset i decreases with b.

Asset prices/returns

In [BVW 14], $\mathscr{D}_b$ is replaced by

$$\mathscr{D}_b^{BVW} = \bar{\rho}\Big(b - \frac{\rho}{\rho + \bar{\rho}}\Big)_+.$$

Note that

$$\mathscr{D}_b < \mathscr{D}_b^{BVW}, \qquad \text{for any } b \in (0,1).$$

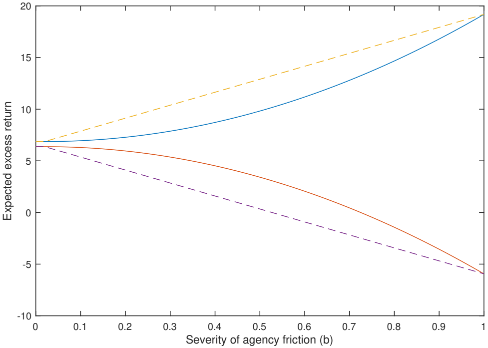


Figure: Solid lines: our result; Dashed lines: [BVW 14].

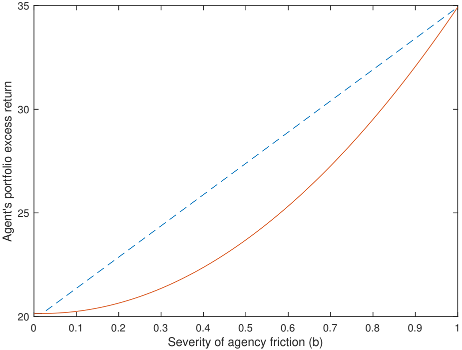
Index and portfolio returns

Excess return of the index

$$\eta'(\mu - r) = r\frac{\rho\bar{\rho}}{\rho + \bar{\rho}} Covar^{\theta,\eta}.$$

Excess return of Manager's portfolio

$$\theta'(\mu - r) = r\frac{\rho\bar{\rho}}{\rho + \bar{\rho}} Var^{\theta} + r\mathscr{D}_b\left(Var^{\theta} - \frac{(Covar^{\theta,\eta})^2}{Var^{\eta}}\right).$$



Optimal contract

$$dF_t = Cdt + \frac{\rho}{\rho+\bar{\rho}} dG_t + \xi(dG_t - \beta^\theta dI_t) + \frac{r}{2}\zeta\, d\langle G - \beta^\theta I, G^\theta - \beta^\theta I\rangle_t$$

- Optimality in a large class of contracts
- **Conjecture:** It is optimal in general.
- $\xi = (b - \frac{\rho}{\rho+\bar{\rho}})_+$, $\zeta = (\rho + \bar{\rho})(b + \xi)(1 - b - \xi)\xi$
- When $b \leq \frac{\rho}{\rho+\bar{\rho}}$, $\xi = \zeta = 0$, only the first two terms show up. The return of the fund is shared between investor and portfolio manager with ratio $\frac{\rho}{\rho+\bar{\rho}}$.

BVW 14 contract corresponds to the two terms in the middle.

- The quadratic variation term is new.
- The term $\langle G - \beta^\theta I, G - \beta^\theta I\rangle$ rewards Manager to take the specific risk of individual stocks, and not only the systematic risk of the index.

Optimal strategy

Manager's vector of optimal holdings is given by

$$Y^* = \frac{1}{r}\frac{1}{\mathcal{C}_b}\Sigma_R^{-1}(\mu - r) + \frac{1}{r}\left(\frac{\rho+\bar{\rho}}{\rho\bar{\rho}}\frac{\mathcal{D}_b}{\mathcal{C}_b}\right)\frac{\eta'(\mu - r)}{Var^{\eta}}\eta, \tag{1}$$

where

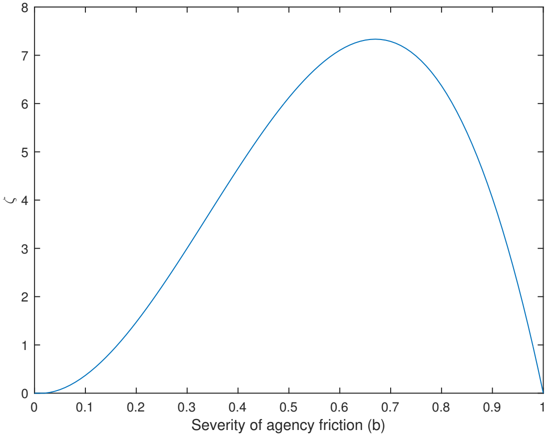
$$\begin{gathered}\mathcal{D}_b = (\rho+\bar{\rho})\big(b - \tfrac{\rho}{\rho+\bar{\rho}}\big)_+^2, \\ \mathcal{C}_b = \tfrac{\rho\bar{\rho}}{\rho+\bar{\rho}} + \mathcal{D}_b.\end{gathered} \tag{2}$$

Optimal contract

When $b \geq \frac{\rho}{\rho+\bar{\rho}}$,

ξ is increasing in b, so as to make Manager to not employ the shirking action.

Dependence of ζ on b:



New contract improves Investor's value

For the asset price in [BVW 14], Investor's value is improved by using the new contract.

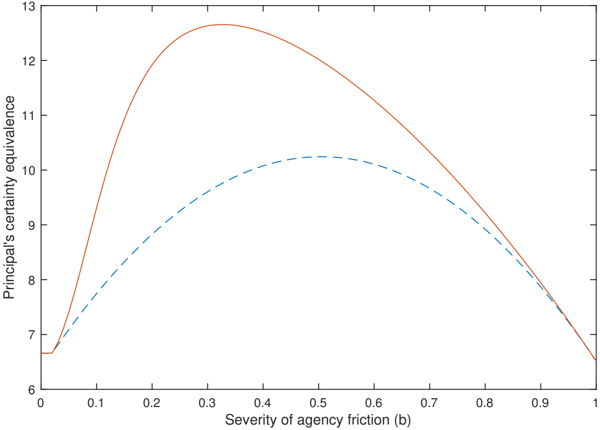


Figure: Solid line: our contract, Dashed line: [BVW 14]

Outline

Admissible contracts: motivation

For any Manager's admissible strategy $\Xi = (\bar{c}, Y, m)$, consider

$$\Xi^t = \{\hat{\Xi} \text{ admissible } \mid \hat{\Xi}_s = \Xi_s, s \in [0, t]\}.$$

Define Manager's *continuation value process* $\bar{\mathcal{V}}(\Xi)$ as

$$\bar{\mathcal{V}}_t(\Xi) = \text{ess sup}_{\Xi^t} \mathbb{E}_t \Big[\int_t^\infty e^{-\bar{\delta}(s-t)} u_A(\bar{c}_s) ds \Big], \quad t \geq 0.$$

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(i) $\partial_{\bar{W}_t} \bar{\mathcal{V}}_t(\Xi) = -r\bar{\rho}\bar{\mathcal{V}}_t(\Xi)$;

(ii) Transversality condition: $\lim_{t\to\infty} \mathbb{E}\big[e^{-\bar{\delta}t}\bar{\mathcal{V}}_t(\Xi)\big] = 0$;

(iii) Martingale principle:

$$\tilde{\mathcal{V}}_t(\Xi) = e^{-\bar{\delta}t}\bar{\mathcal{V}}_t(\Xi) + \int_0^t e^{-\bar{\delta}s} u_A(\bar{c}_s) ds,$$

is a supermartingale for arbitrary admissible strategy Ξ, and is a martingale for the optimal strategy Ξ^*.

Admissible contracts: definition

(Motivated by CPT (2016), (2017))

A contract F is admissible if

1. there exists a constant $\bar{V}_0$,
2. for any Agent's strategy there exist $\mathbb{F}^{G,I}$-adapted processes $Z, U, \Gamma^G, \Gamma^I, \Gamma^{GI}$ such that the process $\bar{V}(\Xi)$, defined via

$$\begin{aligned} d\bar{V}_t(\Xi) = & X_t\Big[(bm_t - \bar{c}_t)dt + Z_t dG_t + U_t dI_t \\ & + \tfrac{1}{2}\Gamma^G_t d\langle G, G\rangle_t + \tfrac{1}{2}\Gamma^I_t d\langle I, I\rangle_t + \Gamma^{GI}_t d\langle G, I\rangle_t\Big] \\ & + \bar{\delta}\bar{V}_t(\Xi)dt - H_t dt, \quad \bar{V}_0(\Xi) = \bar{V}_0, \end{aligned}$$

where $X_t = -r\bar{\rho}\bar{V}_t(\Xi)$ and H is the Hamiltonian

$$\begin{aligned} H = \sup_{\bar{c}, m \geq 0, Y} \Big\{ & u_A(\bar{c}) + X\Big[bm - \bar{c} - Zm + ZY'(\mu - r) + U\eta'(\mu - r) \\ & + \tfrac{1}{2}\Gamma^G Y'\Sigma_R Y + \tfrac{1}{2}\Gamma^I \eta'\Sigma_R \eta + \Gamma^{GI} Y'\Sigma_R \eta\Big]\Big\}, \end{aligned}$$

satisfies $\lim_{t\to\infty} \mathbb{E}\big[e^{-\bar{\delta}t}\bar{V}_t(\Xi)\big] = 0$.

Manager's optimal strategy

Lemma

Given an admissible contract with

$$X > 0, \quad Z \geq b, \quad \textit{and} \quad \Gamma^G < 0,$$

the Manager's optimal strategy is the one maximizing the Hamiltonian,

$$\bar{c}^* = (u_A')^{-1}(X), \quad m^* = 0,$$

$$Y^* + y\eta = -\frac{Z}{\Gamma^G}\Sigma_R^{-1}(\mu - r) - \frac{\Gamma^{GI}}{\Gamma^G}\eta,$$

and we have

$$\bar{V}(\Xi) = \hat{\mathcal{V}}(\Xi).$$

Do we lose on generality?

[CPT 2016, 2016] considered the finite horizon case,

$$
\begin{aligned}
d\bar{V}_t =& X_t \Big[b m_t dt + Z_t dG_t + U_t dI_t \\
& + \frac{1}{2}\Gamma_t^G d\langle G, G\rangle_t + \tfrac{1}{2}\Gamma_t^I d\langle I, I\rangle_t + \Gamma_t^{GI}\langle G, I\rangle_t \Big] - H_t dt.
\end{aligned}
$$

$\bar{V}_T = \mathcal{C}_T$ is the lump-sum compensation paid.

They showed the set of $\mathcal{C}$ that can be represented as $\bar{V}_T$ is dense in the set of all (reasonable) contracts. Hence, there is no loss of generality in their framework.

Their proof is based on the 2BSDE theory, e.g., [Soner-Touzi-Zhang 2011,12,13].

Conjecture: A similar result holds for the infinite horizon case. (Work in progress by Lin, Ren, and Touzi.)

Representation of admissible contracts

Lemma

An admissible contract F can be represented as

$$\begin{aligned} dF_t =& Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G \, d\langle G, G\rangle_t + \tfrac{1}{2}\Gamma_t^I \, d\langle I, I\rangle_t + \Gamma_t^{GI} \, d\langle G, I\rangle_t \\ &+ \tfrac{1}{2} r\bar{\rho} \, d\langle Z\cdot G + U\cdot I, Z\cdot G + U\cdot I\rangle_t - \left(\tfrac{\bar{\delta}}{r\bar{\rho}} + \bar{H}_t\right) dt, \end{aligned}$$

where $Z \cdot G = \int_0^{\cdot} Z_s dG_s$ *and*

$$\begin{aligned} \bar{H}_t =& \tfrac{1}{\bar{\rho}} \log(-r\bar{\rho}\bar{V}_0) - \tfrac{1}{\bar{\rho}} + (Z_t Y_t^* + U_t \eta)'(\mu_t - r) \\ &+ \tfrac{1}{2}\Gamma_t^G (Y_t^*)'\Sigma_R Y_t^* + \tfrac{1}{2}\Gamma_t^I \eta'\Sigma_R \eta + \Gamma_t^{GI} (Y_t^*)'\Sigma_R \eta. \end{aligned}$$

In particular, F *is adapted to* $\mathbb{F}^{G,I}$ *(as it should be).*

Investor's problem

1. Guess Investor's value function

$$V(w) = Ke^{-r\rho w},$$

2. Treat $Z, U, \Gamma^G, \Gamma^{GI}$ as Investor's control variables.

3. Work the with HJB equation satisfied by V.

Conclusion

- We find an asset pricing equilibrium with the contract optimal in a large class. (Maybe the largest.)
- Price/return distortion less sensitive to agency frictions.
- The contract also based on the second order variations.

Future work:

- Square root, CIR dividend processes

Happy birthday Yannis!