Fluctuations of interacting particle systems

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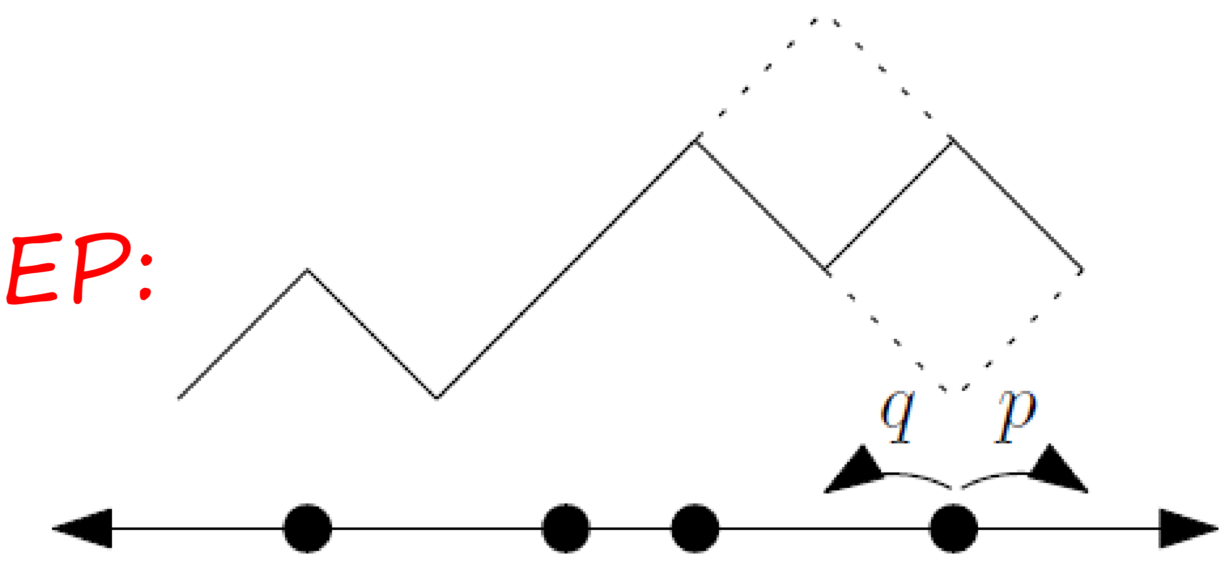
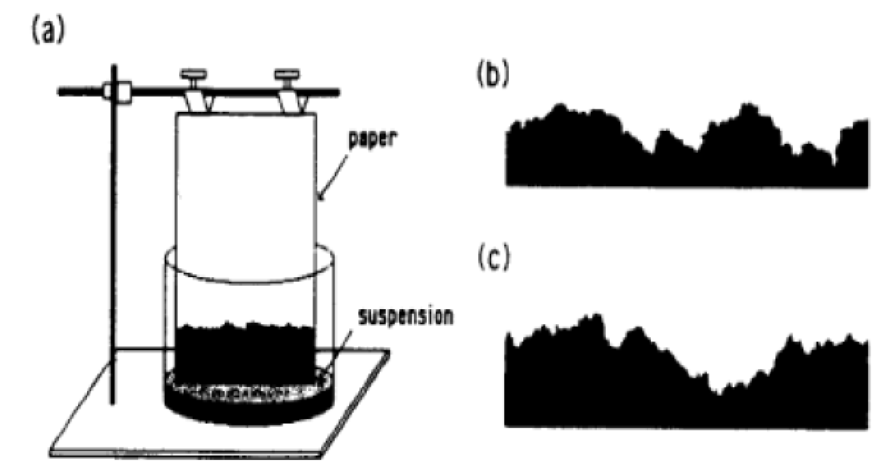
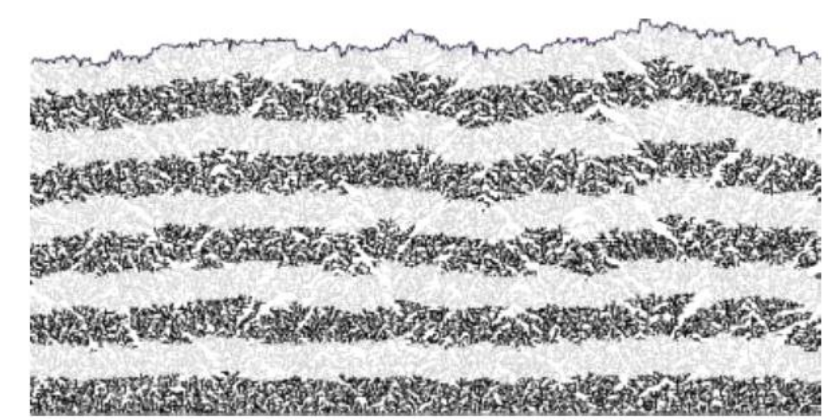
Interacting particle systems & ASEP

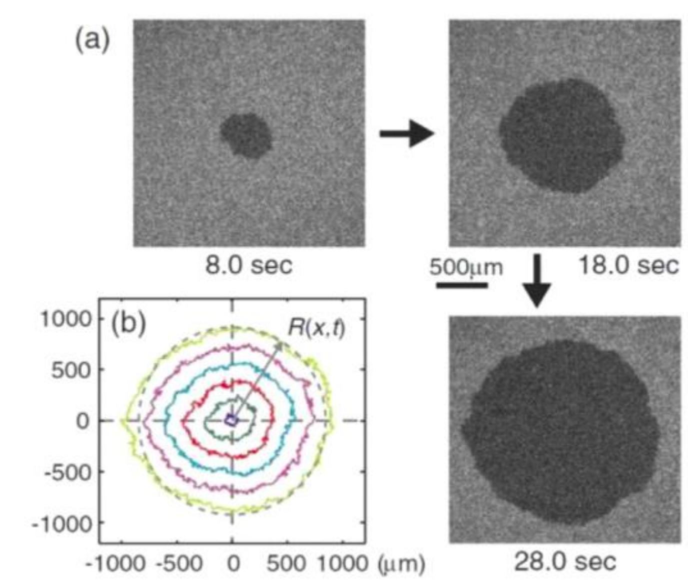
In one dimension these model mass transport, traffic, growth...

ASEP:









Some key considerations and questions:

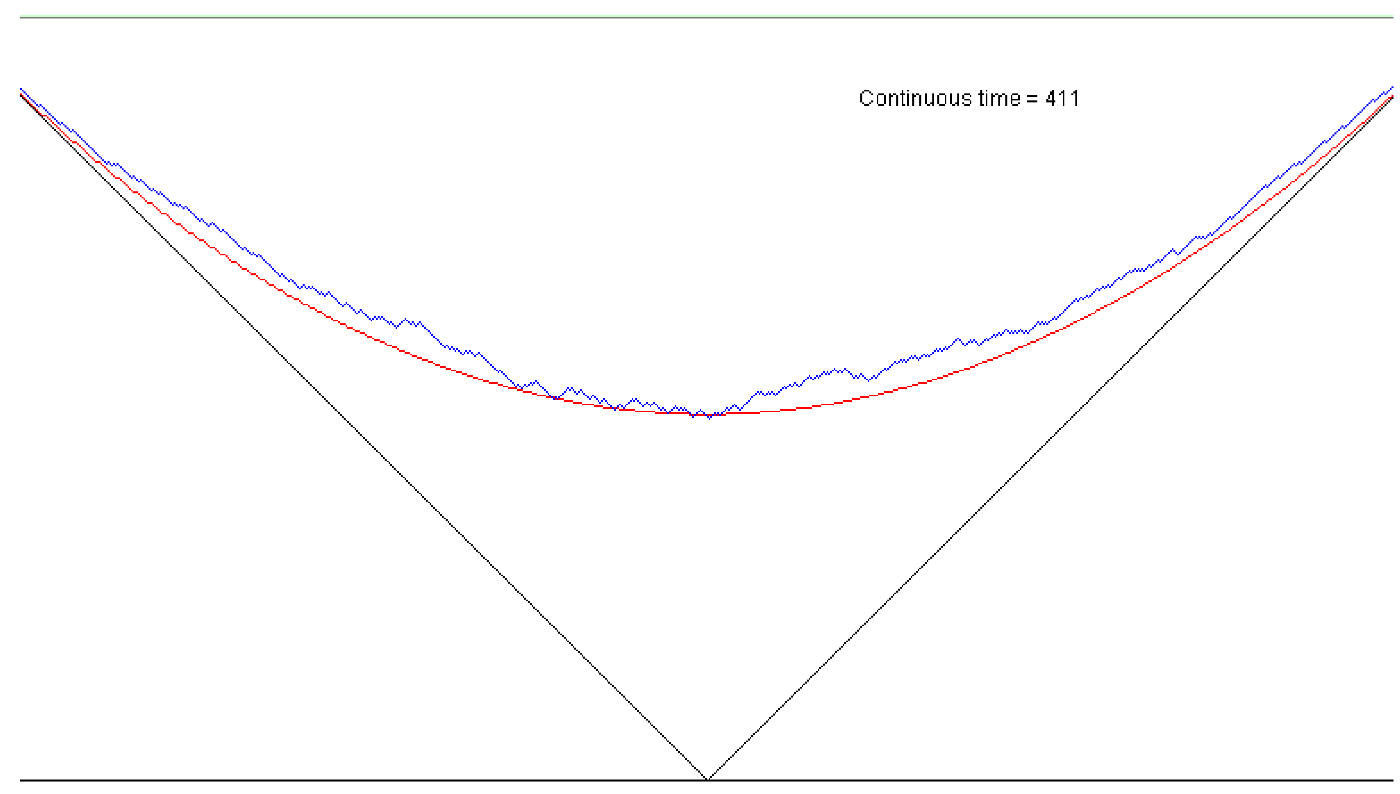
- Invariant measures and expectations;
- LLN / PDE (hydrodynamic) limits;
- Large deviation principals;
- Fluctuation and stochastic PDEs limits.

We'll focus ASEP, which predicts behavior of the full class.

TASEP (q=0 ASEP)

Solvable due to connections with Schur polynomials, free Fermions, determinantal point processes, biorthogonal ensembles...

[Johansson '99, Prahofer-Spohn '02]:
In long time, TASEP with step initial data has height fluctuations which grow like time^1/3 with correlations in the time^2/3 transversal scale and Airy process multipoint distributions.



Work since has extended to general initial data and developed the full space time limit of TASEP (called the KPZ fixed point).

Universality is out of reach, but we can test on other solvable models.

ASEP (q<p), KPZ equation

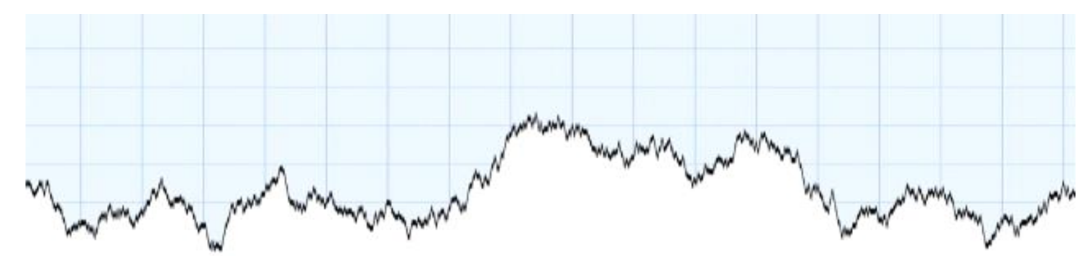
ASEP is solvable via Bethe ansatz and Hall-Littlewood polynomials.

[Tracy-Widom '09]: In long time, ASEP with step initial data has height fluctuations exponent 1/3 and limiting GUE one-point distribution.

[C-Dimitrov '17]: ASEP has transversal scaling exponent 2/3 with a limiting spatial process which is absolutely continuous w.r.t Brownian motion.

space-time white noise

$$\partial_t h(t,x) = \frac{1}{2}\partial_{xx} h(t,x) + \frac{1}{2}\left(\partial_x h(t,x)\right)^2 + \xi(t,x)$$

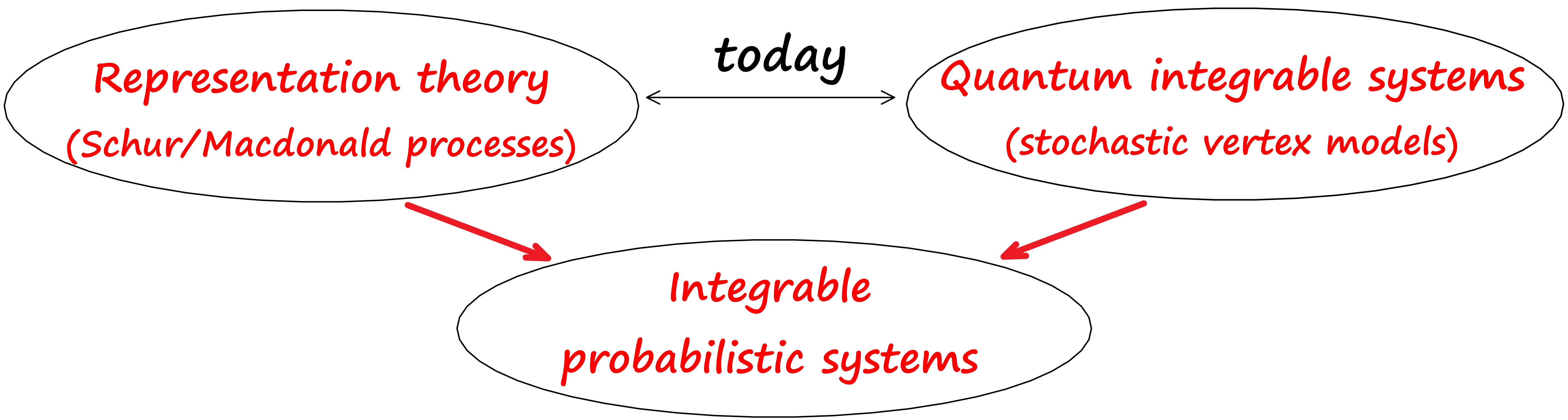


- Kardar-Parisi-Zhang (KPZ) SPDE: [Amir-C-Quastel '11] proved 1/3 exponent and GUE limit; [C-Hammond '13] proved 2/3 exponent and Brownian abs. cont.
- Another ASEP limit is to Brownian motions with skew reflection. ASEP methods should survive that limit ([Sasamoto-Spohn '15] prove 1/3; 2/3 not yet proved).

Integrable probability in a nutshell

Study scaling and statistics of complex random systems through exactly solvable examples which predict larger universality class.

These special systems come from algebraic structures:



Connecting these two sides yields new tools in studying models such as tilings, stochastic six vertex model and ASEP.

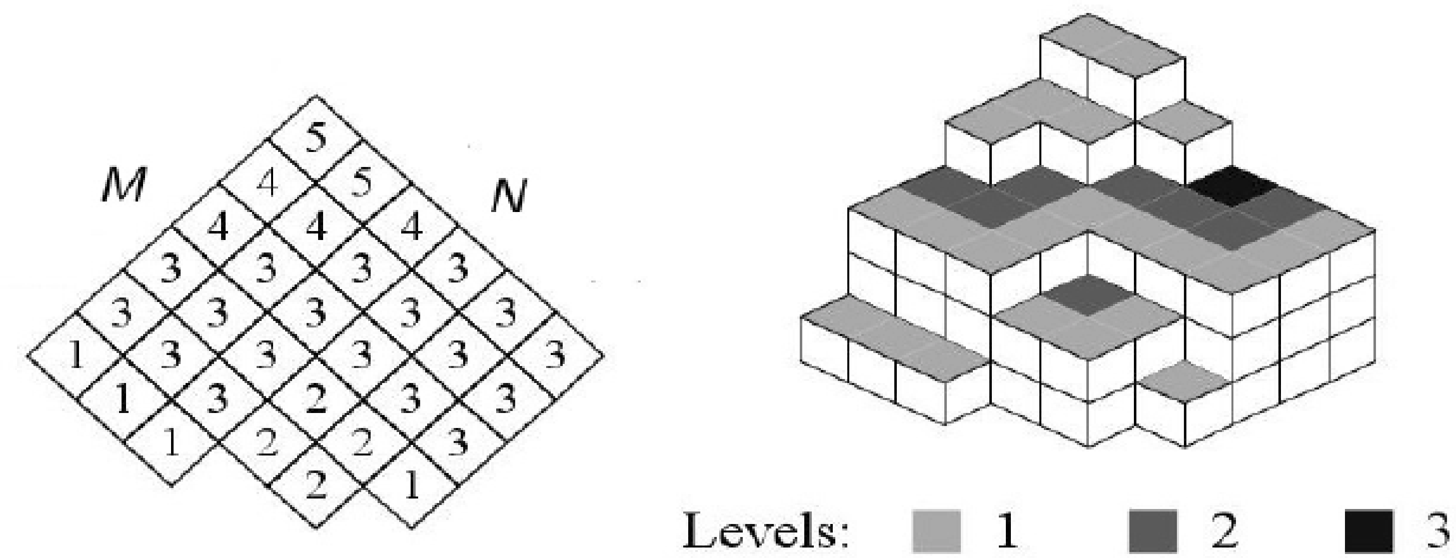
Tiling

We consider a measure on plane partitions (equivalently rhombus tilings, dimers, or 3d Young diagrams) determined by ζ and t as:

$$\mathrm{Prob}(\pi) = \zeta^{\mathrm{diag}(\pi)} A_\pi(t)$$

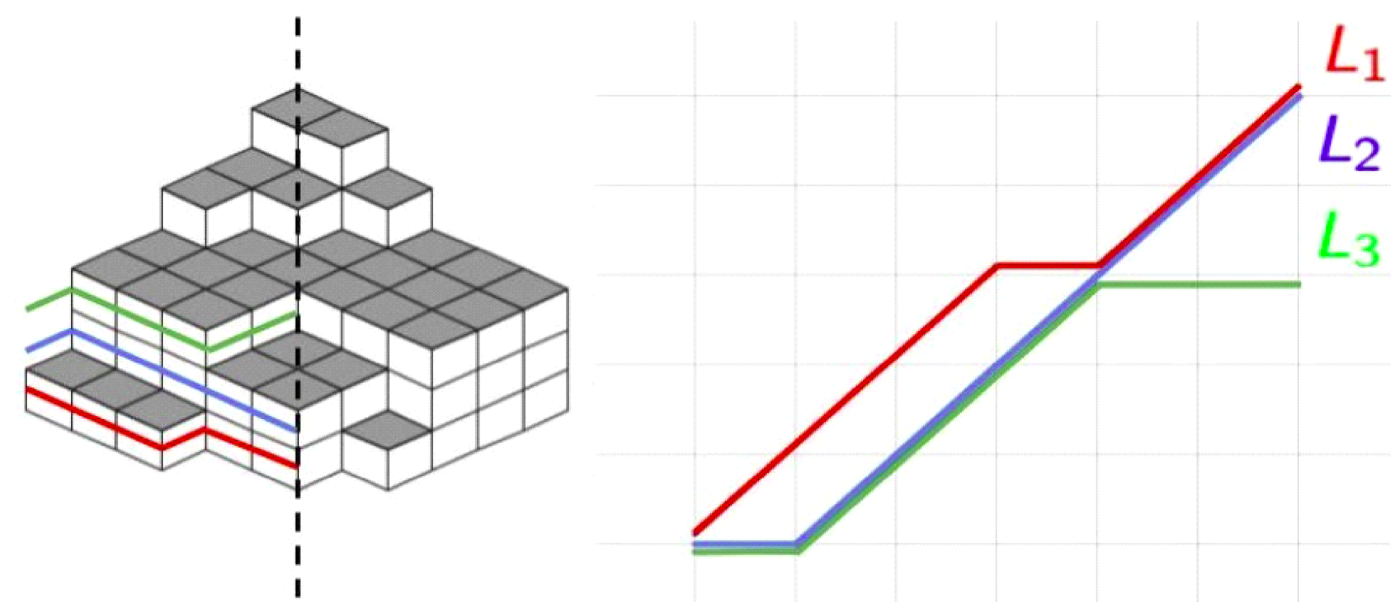
where $\mathrm{diag}(\pi) = \sum_i \pi_{i,i}$ and

$$A_\pi(t) = \prod_{(i,j)\in\mathrm{supp}(\pi)} (1 - t^{\mathrm{level}}).$$



Eg. $\mathrm{diag}(\pi) = 16$ $\quad A_\pi(t) = (1-t)^7(1-t^2)^3(1-t^3)$

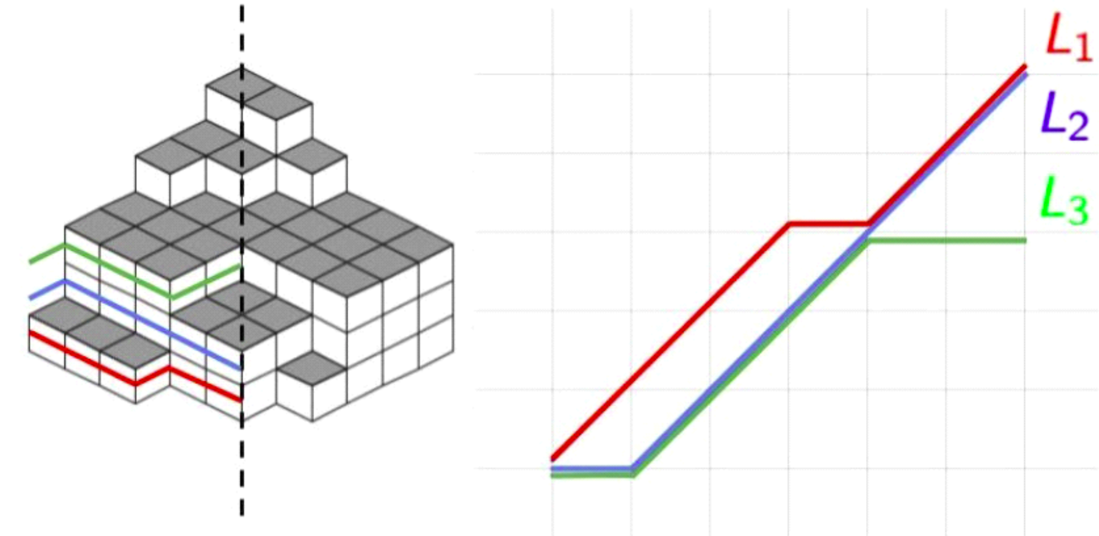
We associate an ensemble of non-crossing level lines which we call the Hall-Littlewood line ensemble.



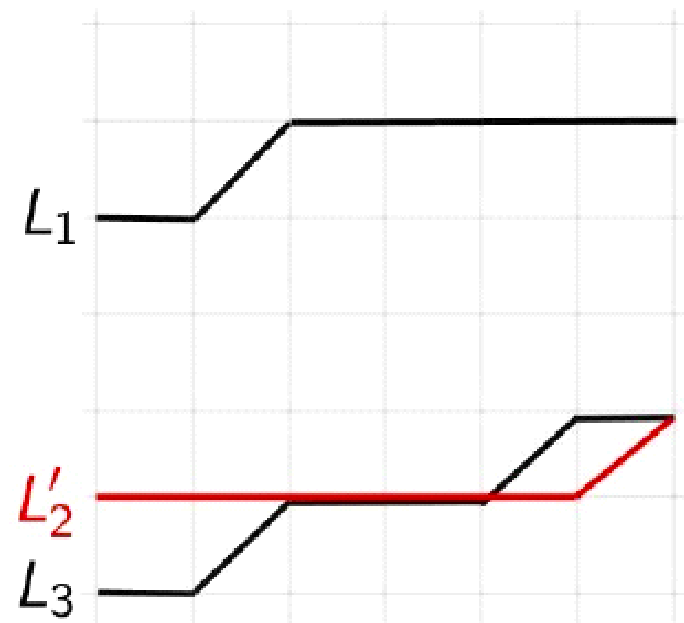
Generalizes Schur process / tiling of [Okounkov-Reshetikhin '01].

Hall-Littlewood Gibbs property

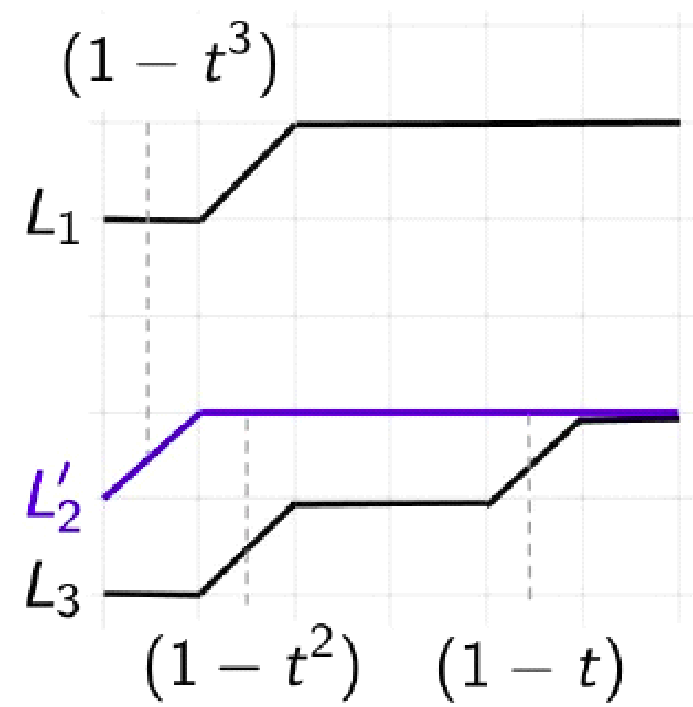
The Hall-Littlewood line ensemble enjoys a Gibbs resampling property.



Given curve above and below, the law of middle curve is (uniform) x (weight depending locally on the derivative of height differences).

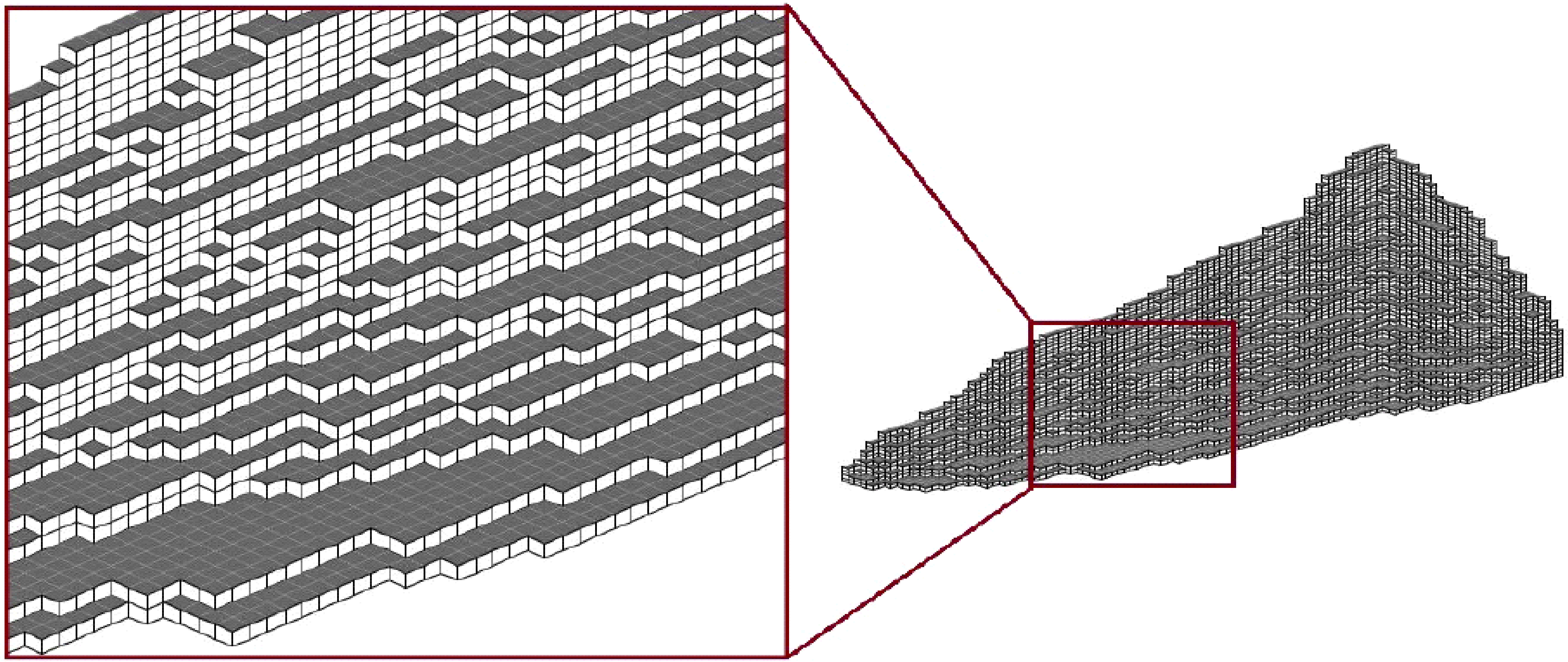


$W_t(L_2', L_1, L_3) = 0$



$W_t(L_2', L_1, L_3) = (1-t)(1-t^2)(1-t^3)$

Tightness



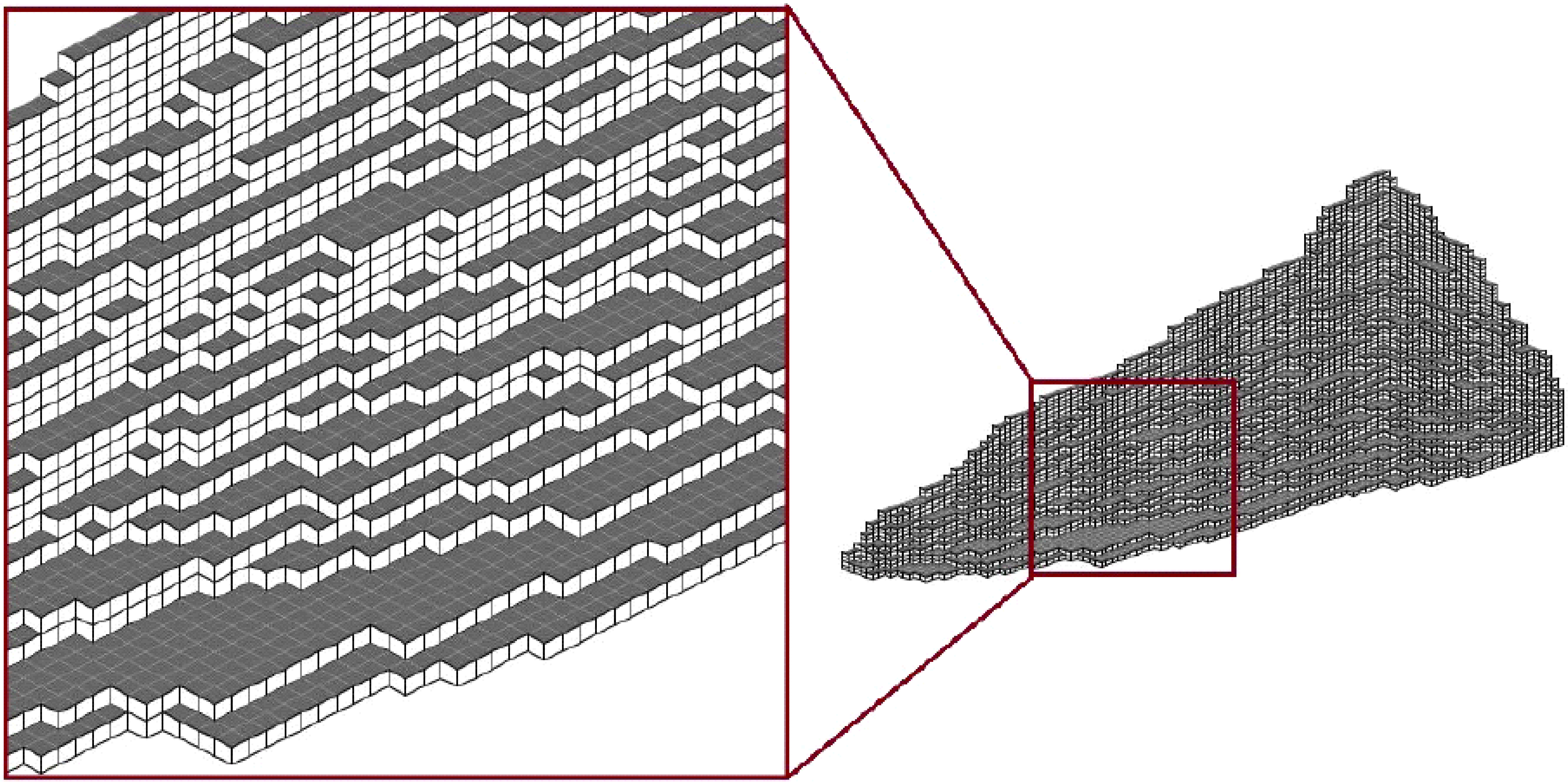
[C-Dimitrov '17] (building on [C-Hammond '11,'13]) show that one point tightness of the top curve (base of the tiling) implies spatial tightness for the full edge ensemble under diffusive scaling.

Caution: HL Gibbs property does not enjoy monotone coupling (like non-intersecting random walks / BM) so we had to develop weaker forms of monotonicity.

Tiling limit shape?

Taking M, N large seems to yields a limit shape -- what is it?

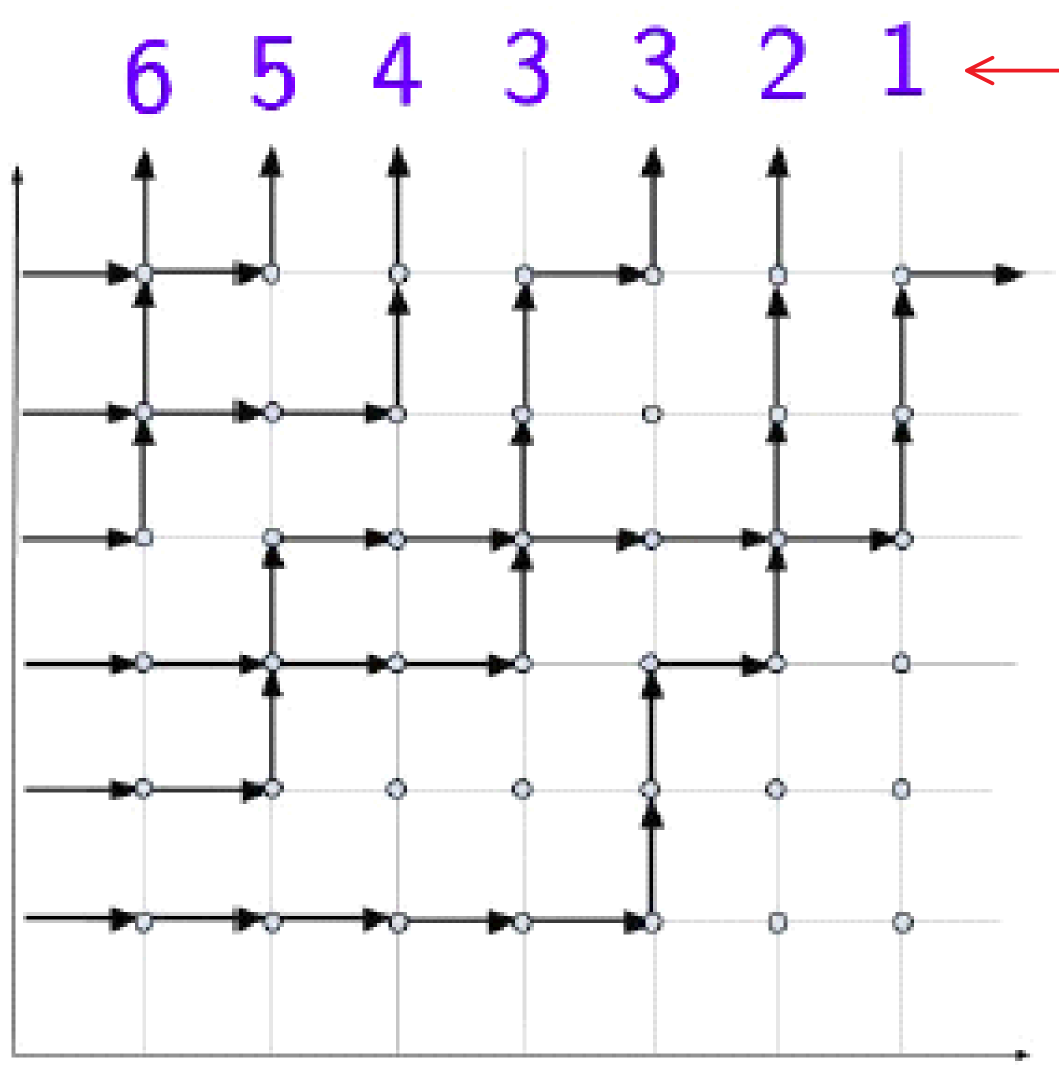
We prove edge fluctuation exponent 1/3, transversal exponent 2/3.



S6V

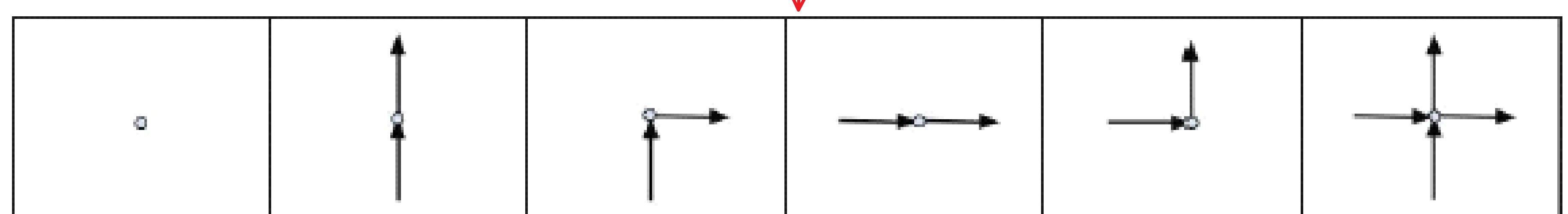
Stochastic six vertex model [Gwa-Spohn '93], [Borodin-C-Gorin '15]

(Gauge-transform of the a,b,c model where weights sum for fixed input to 1.)



Height function $h(x, N)$ records number of arrows at or to the right of a given location.

Stochastic weights

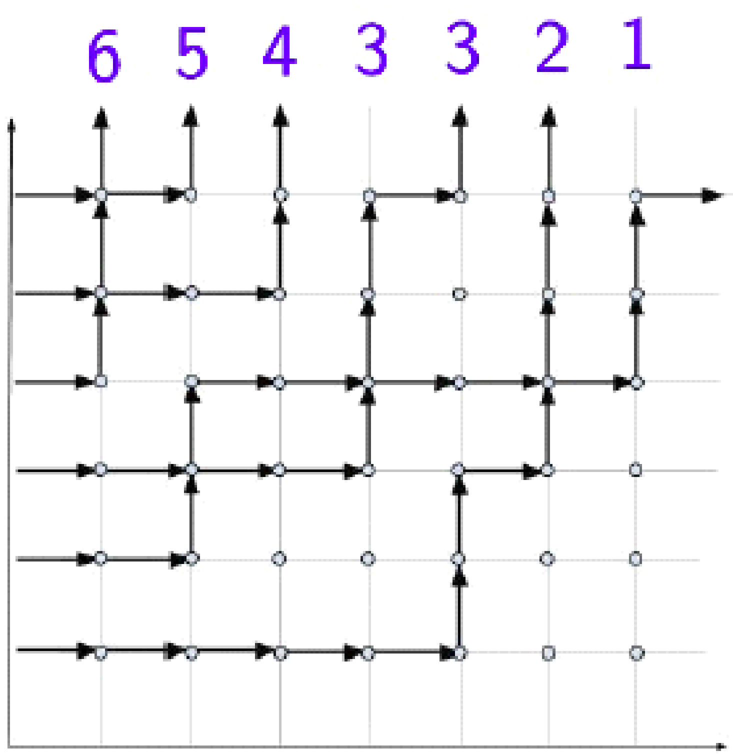


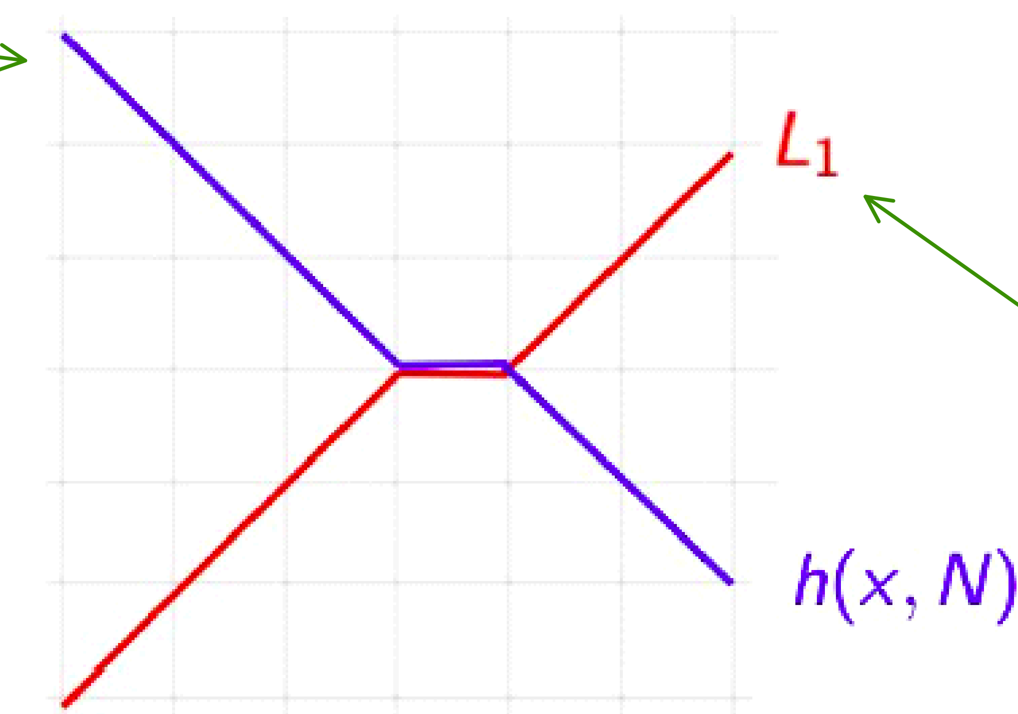
1	b_1	$1-b_1$	b_2	$1-b_2$	1

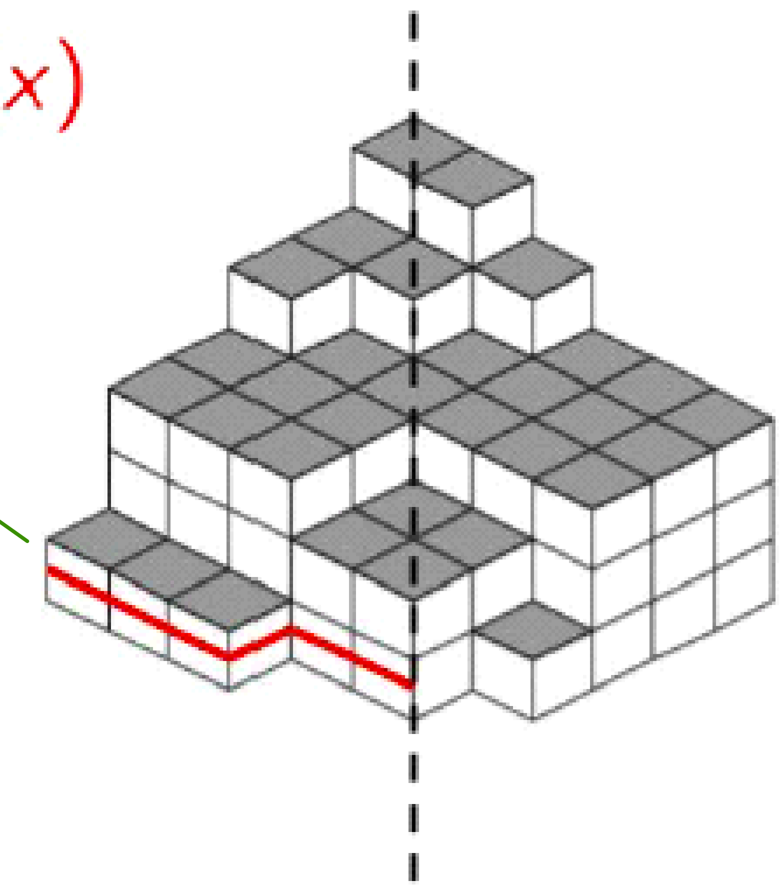
Tiling <--> S6V

[Borodin-Bufetov-Wheeler '17] relate these two models so that

$h(x,N)$ equals in law $N - L_1(x)$







With $b_1 = t \cdot \frac{1-\zeta}{1-t\zeta}$

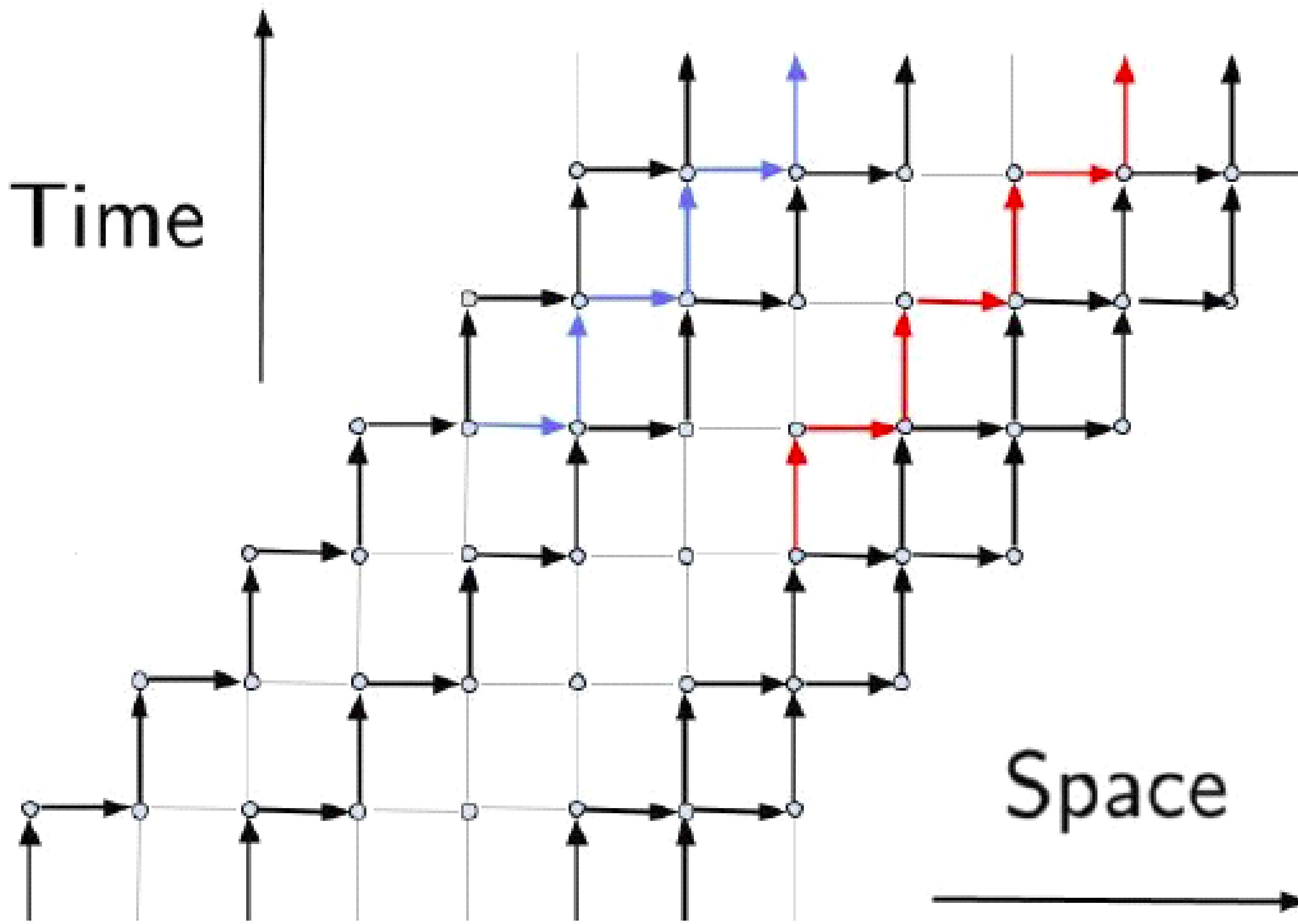
and $b_2 = \frac{1-\zeta}{1-t\zeta}$

With $\mathrm{Prob}(\pi) = \zeta^{\mathrm{diag}(\pi)} A_\pi(t)$

Proved by relating tiling to a vertex model and using Yang-Baxter.

S6V -> ASEP

Taking $b_1 = \epsilon q$, $b_2 = \epsilon p$, $N = \epsilon^{-1} T$, $x = \epsilon^{-1} T + \tilde{x}$, and ϵ to 0 the S6V height function converges to that of ASEP.



This is just like how the a,b,c 6 vertex model goes to XXZ spin chain

Overview of connections

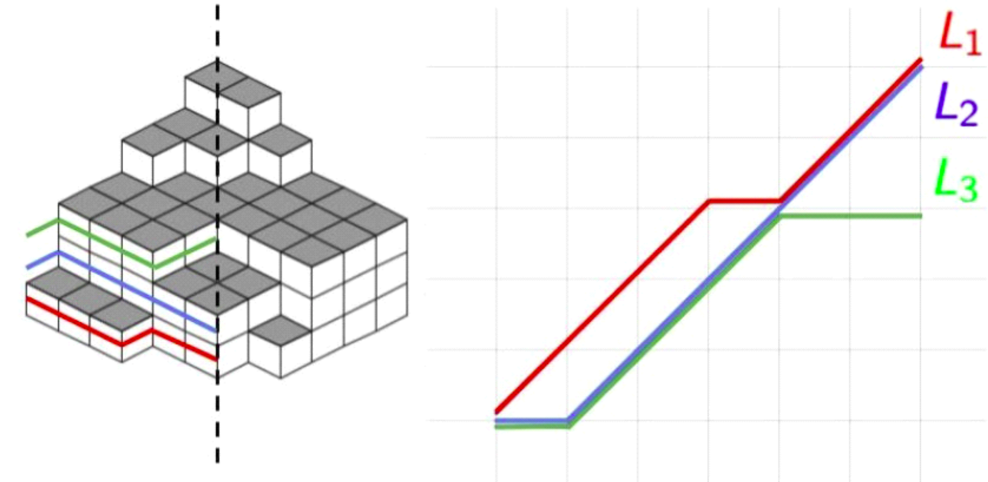
Hall-Littlewood process

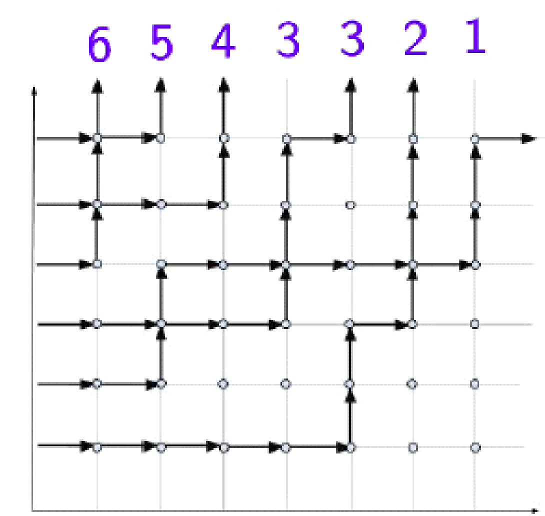
$\updownarrow$

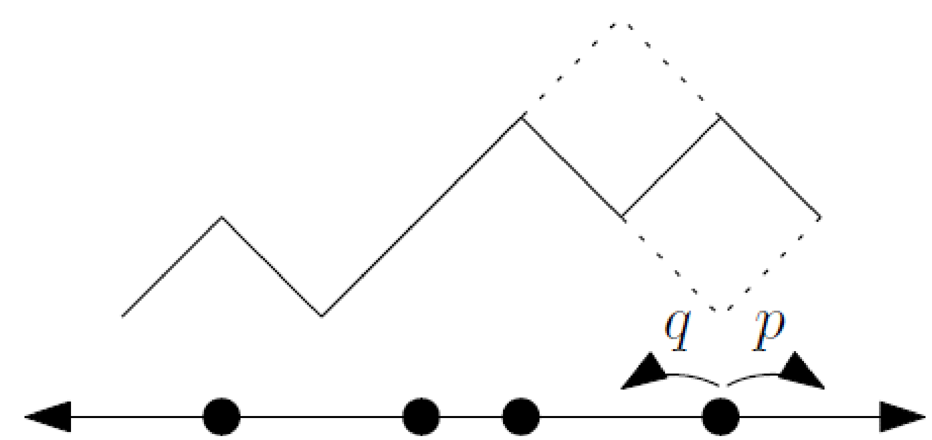
Stochastic six vertex

$\downarrow$

ASEP





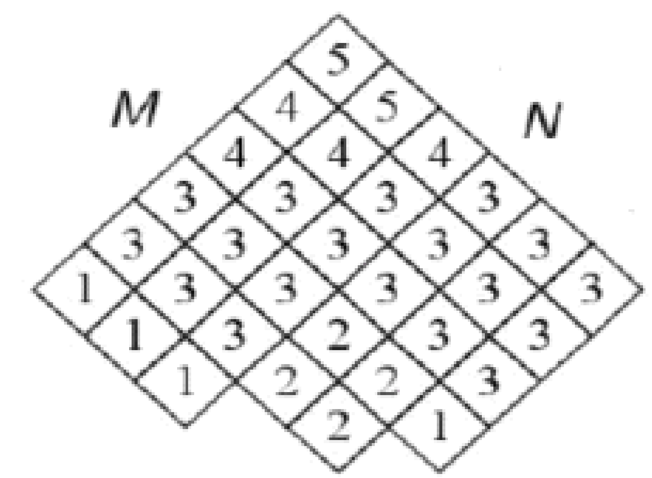


It remains for us to prove time^1/3 edge fluctuation, and tiling<-->S6V relation

time^1/3 proof (via Macdonald processes)

Recast tiling measure as Hall-Littlewood process on sequences of interlacing partitions $\bar{\lambda} = \emptyset \prec \lambda^{(1)} \prec \lambda^{(2)} \cdots \prec \lambda^{(M)} \succ \cdots \succ \lambda^{(M+N)} \succ \emptyset$:

$$\mathbb{P}(\bar{\lambda}) = \frac{1}{Z} \prod_{i=1}^{M} P_{\lambda^{(i)}/\lambda^{(i-1)}}(a) \prod_{i=M}^{M+N} Q_{\lambda^{(i)}/\lambda^{(i+1)}}(b)$$



where $ab = \zeta$. The one variable skew Hall-Littlewood polynomials are

$$P_{\lambda/\mu}(a) := a^{|\lambda|-|\mu|} \mathbf{1}_{\lambda \succ \mu} \prod_{j=1}^{\infty} \left(1 - \mathbf{1}_{\Delta(\mu,j)-\Delta(\lambda,j)}\right) t^{\Delta(\mu,j)}$$

with $\Delta(\mu, j) = \mu'_j - \mu'_{j+1}$ and $Q_{\lambda/\mu}$ defined similarly.

The quantity of interest is the length of $\lambda^{(M)}$ (or first row of its transpose).

time^1/3 proof (via Macdonald processes)

The marginal distribution of $\lambda^{(M)}$ is a Hall-Littlewood measure

$$\mathbb{P}(\lambda) = \frac{1}{Z(\vec{a},\vec{b})} P_\lambda(\vec{a}) Q_\lambda(\vec{b})$$

where the Hall-Littlewood symmetric polynomials are defined via

$$P_\lambda(a_1,\ldots,a_M) := \sum_{\lambda^{(1)}\prec\cdots\prec\lambda^{(M)}=\lambda} P_{\lambda^{(1)}}(a_1) P_{\lambda^{(2)}/\lambda^{(1)}}(a_2)\cdots P_{\lambda^{(M)}/\lambda^{(M-1)}}(a_M)$$

$$= \frac{1}{v_\lambda(t)} \sum_{\sigma\in S_M} \sigma\left(a_1^{\lambda_1}\cdots a_M^{\lambda_M} \prod_{i<j} \frac{a_i - t a_j}{a_i - a_j}\right)$$

$$v_\lambda(t) = \prod_{i\geq 0} [m_i]_t$$

$$[n]_t = \frac{(t;t)_n}{(1-t)^n}$$

$$\lambda = 0^{m_0} 1^{m_1} 2^{m_2}\cdots$$

Hall-Littlewood polynomials $P_\lambda(\vec{a};0,t)$ are special cases of the Macdonald polynomials $P_\lambda(\vec{a};q,t)$ (and generalize Schur $P_\lambda(\vec{a};t,t) = s_\lambda(\vec{a})$).

Macdonald processes

Macdonald processes $q,t\in[0,1)$

Ruijsenaars–Macdonald system

Representations of Double Affine Hecke Algebras

q-Whittaker processes $t=0$

q-TASEP, 2d dynamics

q-deformed quantum Toda lattice

Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$

Hall-Littlewood processes $q=0$

Random matrices over finite fields

Spherical functions for p-adic groups

General β RMT $t=q^{\beta/2}\to 1$

Random matrices over $\mathbb{R},\mathbb{C},\mathbb{H}$

Calogero-Sutherland, Jack polynomials

Spherical functions for Riem. Symm. Sp.

Whittaker processes $t=0$, $q\to 1$

Directed polymers and their hierarchies

Quantum Toda lattice, repr. of $GL(n,\mathbb{R})$

Kingman partition structures

Cycles of random permutations $q=0$

Poisson-Dirichlet distributions $t=1$

Schur processes $q=t$

Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE

Characters of symmetric, unitary groups

Hall-Littlewood expectations via Schur processes

The Macdonald Cauchy identity yields the normalizing constant

$$Z(\vec{a},\vec{b};q,t)=\sum_{\lambda}P_\lambda(\vec{a};q,t)Q_\lambda(\vec{a};q,t)=\prod_{i,j}\frac{(ta_ib_j;q)_\infty}{(a_ib_j;q)_\infty}$$

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-q^k a)$$

Macdonald difference operators act diagonally on the polynomials:

$$D_n^u P_\lambda(x_1,\ldots,x_n)=\prod_{i=1}^{n}\left(1-uq^{\lambda_i}t^{n-i}\right)P_\lambda(x_1,\ldots,x_n)$$

$$D_n^u=\sum_{I\subseteq\{1,\ldots,n\}}(-u)^{|I|}t^{|I|(|I|-1)/2}\prod_{i\in I,j\notin I}\frac{tx_i-x_j}{x_i-x_j}\prod_{i\in I}T_{q,x_i}$$

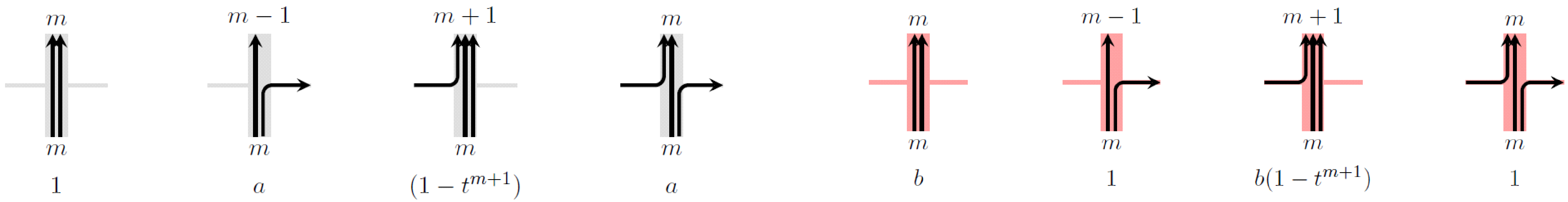
Recipe to compute expectations: $\dfrac{D_n^u Z(\vec{a},\vec{b};q,t)}{Z(\vec{a},\vec{b};q,t)}=\mathbb{E}\left[\prod_{i=1}^{n}\left(1-uq^{\lambda_i}t^{n-i}\right)\right]$

Easy to see the LHS is q-independent (since $\dfrac{Z(a,\vec{b};q,t)}{Z(a,\vec{b};q,t)}=\prod_j\dfrac{1-ab_j}{1-tab_j}$) hence

$$\mathbb{E}^{\mathrm{Schur}}\left[\prod_{i=1}^{n}\left(1-ut^{\lambda_i+n-i}\right)\right]=(u;t)_\infty\mathbb{E}^{\mathrm{HL}}\left[\frac{1}{(ut^{n-\lambda'_1};t)_\infty}\right]$$

reducing our problem to well-known Schur asymptotics.

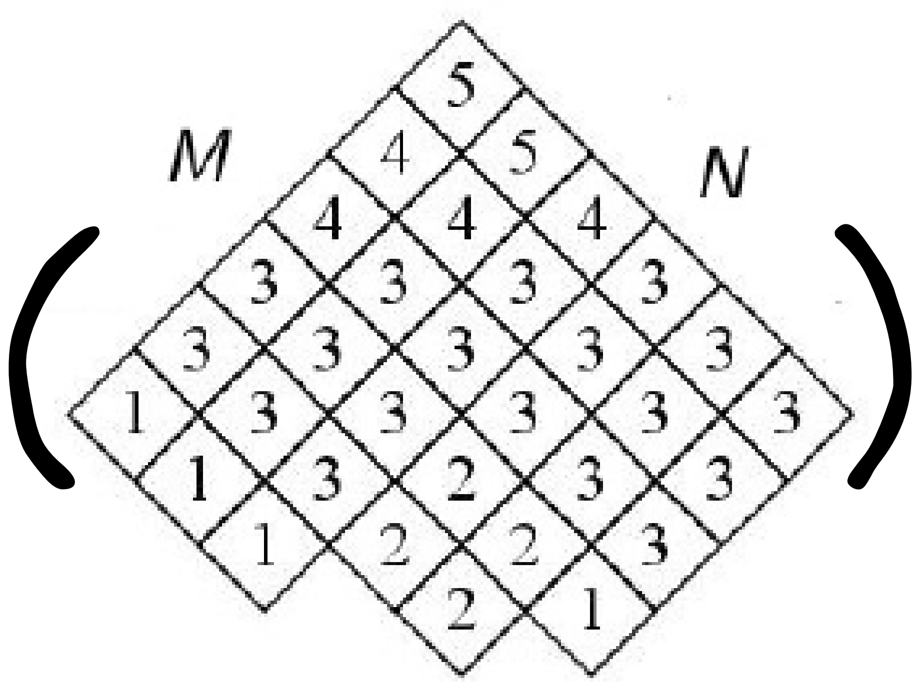
t-Boson vertex model



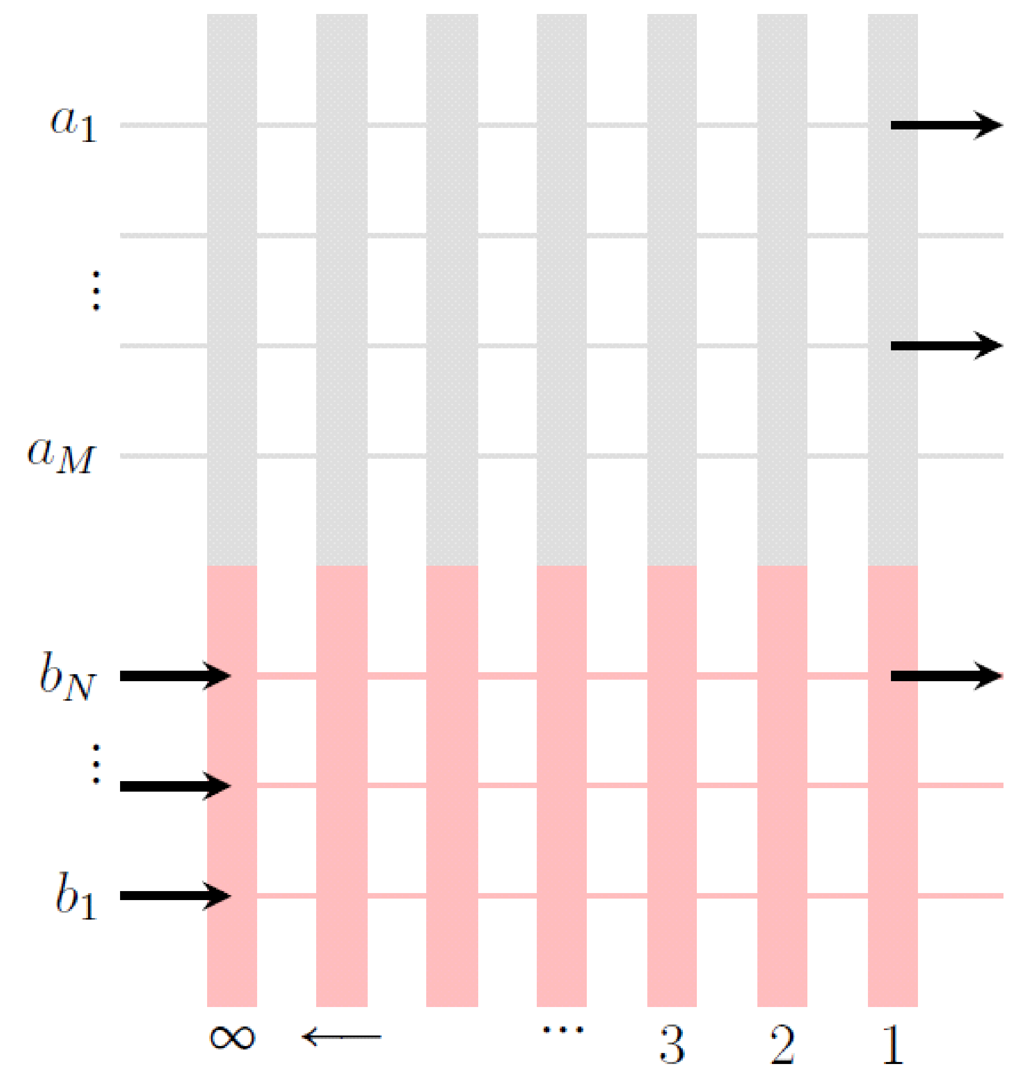
Plane partition (tiling) a formed by increasing, then decreasing interlacing partitions. t-Boson weights induce a measure on such a sequence.

Setting $a_i b_j = \zeta$ we get back our original measure.

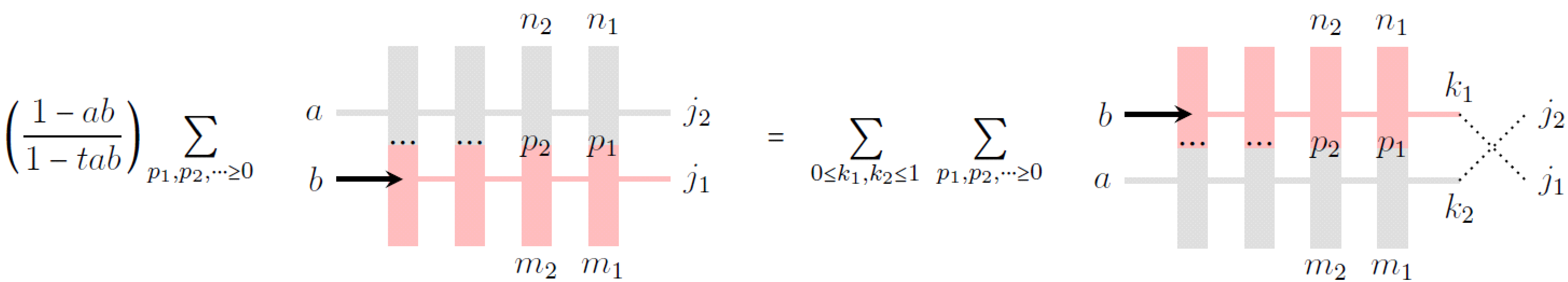
Law of



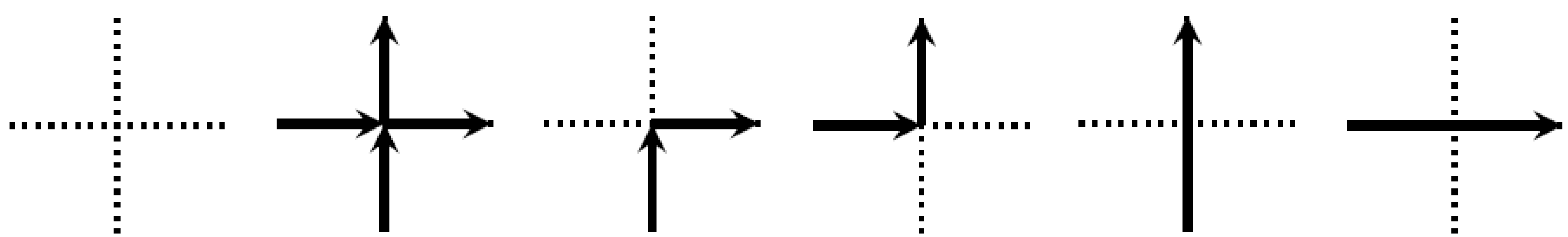
$$= \prod_{i=1}^{M} \prod_{j=1}^{N} \left(\frac{1 - a_i b_j}{1 - t a_i b_j} \right) \times$$



Yang-Baxter equation



The sum is over all internal vertices and on the right is a vertex from the S6V model (rotated 45 degrees) with weights:



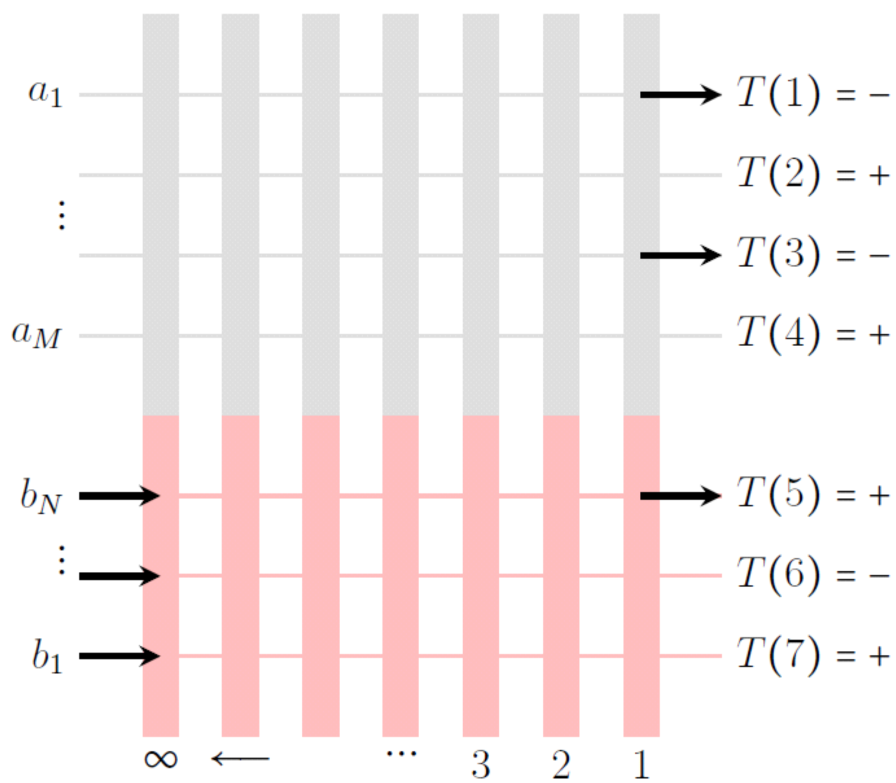
$$1 \qquad 1 \qquad \frac{1-t}{1-tab} \qquad \frac{(1-t)ab}{1-tab} \qquad \frac{t(1-ab)}{1-tab} \qquad \frac{1-ab}{1-tab}$$

Follows single vertex t-Boson YBE by tensoring and taking a limit.

Yang-Baxter equation

Using the YBE to switch the red and grey rows

$$\prod_{i=1}^{M}\prod_{j=1}^{N}\left(\frac{1-a_ib_j}{1-ta_ib_j}\right)\times$$



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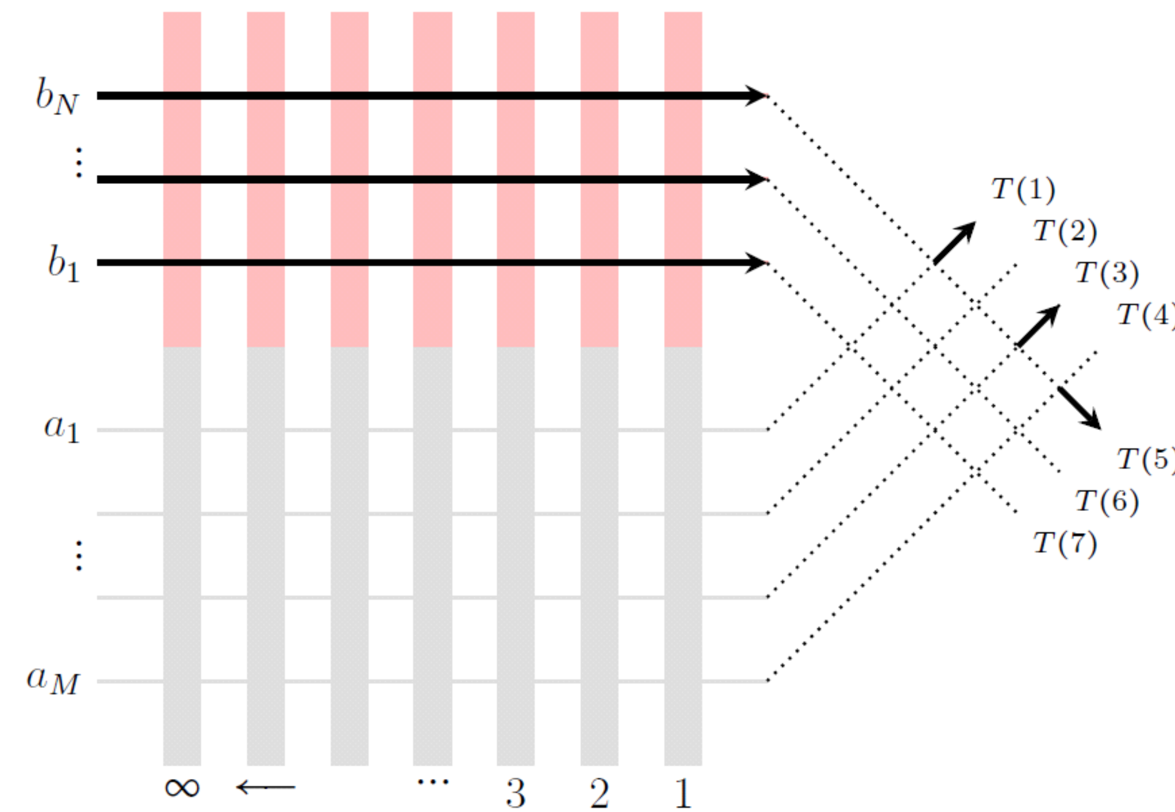
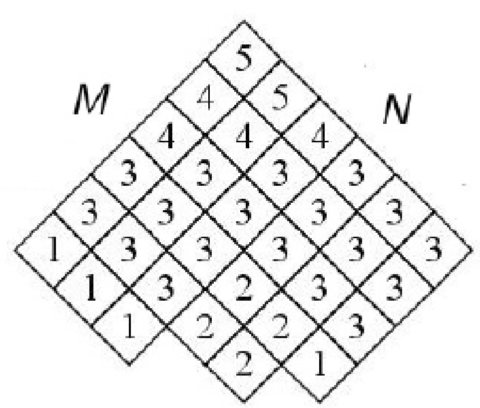
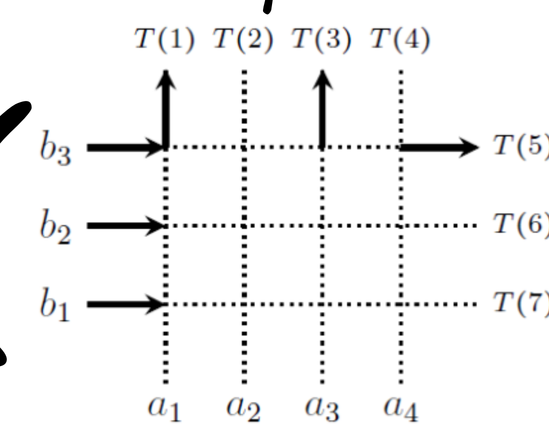


relates law of the tiling base to that of the S6V output arrows.

Law of the base (



) = Law of output (



)

In half space case, have to additionally use "reflection equations".

Summary

- Relate S6V height function to "Hall-Littlewood" tiling base.
  The tiling is a special case of Macdonald processes at q=0.
- Using properties of Macdonald / Hall-Littlewood / Schur symmetric functions we compute certain expectations explicitly and perform one-point asymptotics.
- Using the tiling's Gibbs property, we can extend the one-point 1/3 exponent tightness to the transversal 2/3 exponent.
- Both models admit limits to ASEP and the KPZ equation and hence this provides a means to study those models too.
- **Some questions: Tiling limit shape? Asymptotics for more general boundary rates? Two-sided open ASEP? Higher spin models?**