Qualitative properties of optimal portfolios in log-normal markets

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References

Work in progress

- Temporal and spatial properties of optimal portfolios in log-normal markets (with S. Kallblad)
- Complete monotonicity and marginal utilities (with S. Kallblad)
- The optimal wealth process in log-normal markets (with P. Monin)

The classical Merton problem

The classical Merton problem

- $(\Omega, \mathcal{F}, \mathbb{P})$; W standard Brownian motion

- Traded securities

$$\begin{cases} dS_t = \mu S_t\, dt + \sigma S_t\, dW_t & , \quad S_0 > 0 \\ dB_t = 0 & , \quad B_0 = 1 \end{cases}$$

- Self-financing strategies π_t^0 (bond allocation), π_t (stock allocation)

- Value of allocation $X_t = \pi_t^0 + \pi_t$

$$dX_t = \sigma \pi_t (\lambda\, dt + dW_t) \; ; \quad \lambda = \frac{\mu}{\sigma}$$

Value function

- Trading horizon $[0, T]$, $T < \infty$

- Utility function at T : $U(x)$, $x \geq 0$

- Value function

$$V(x,t) = \sup_{\mathcal{A}} E_{\mathbb{P}}\big(U(X_T)/X_t = x\big)$$

- Set of admissible strategies $\mathcal{A}$

$$\mathcal{A} = \left\{ \pi : \pi_s \in \mathcal{F}_s \ , \ E_{\mathbb{P}} \int_t^T \pi_s^2 \, ds < +\infty \ , \ X^{\pi} \geq 0 \ , \ \text{a.e.} \right\}$$

Optimality and HJB equation

- The value function $\quad V : [0,\infty) \times [0,T] \to [0,\infty)$

$$\text{(HJB)} \quad \begin{cases} V_t + \max_{\pi} \left(\frac{1}{2}\sigma^2 \pi^2 V_{xx} + \mu \pi V_x \right) = 0 \\ V(x,T) = U(x) \end{cases}$$

- Optimal feedback controls

$$\pi^*(x,t) = -\frac{\lambda}{\sigma} \, \frac{V_x(x,t)}{V_{xx}(x,t)}$$

- Optimal wealth process

$$dX_s^* = \mu \pi^*(X_s, s)\, ds + \sigma \pi^*(X_s, s)\, dW_s \; ; \quad X_t = x$$

- Optimal allocations : $\quad \pi_s^{0,*} = X_s^* - \pi_s^*$ (bond), $\quad \pi_s^* = \pi^*(X_s^*, s)$ (stock)

Questions

The optimal feedback portfolio and investment weight are given by

$$\pi^*(x,t;T) = \frac{\lambda}{\sigma} r(x,t;T) \quad \text{and} \quad w^*(x,t;T) = \frac{\lambda}{\sigma}\,\frac{r(x,t;T)}{x},$$

where r is the local risk tolerance function,

$$r(x,t;T) = -\frac{V_x(x,t;T)}{V_{xx}(x,t;T)}$$

We want to investigate for $\pi^*(x,t;T)$, $w^*(x,t;T)$ and $r(x,t;T)$

- Spatial monotonicity
- Spatial concavity/convexity
- Temporal monotonicity
- Sensitivities w.r.t. market parameters and horizon (portfolio greeks)

Fundamental Question

Which properties, qualitative and structural, of quantities prescribed at T (e.g. risk aversion, risk tolerance, utility, marginal utility, inverse marginal utility, prudence,...) are propagated to the analogous quantities at previous trading times?

Previous work

- Spatial monotonicity (Borell; same model)
- Time monotonicity (Gollier; discrete time)
- Rich body of work in one-period models (Arrow, Ross, Kimball, Mossin, Roll, Pratt,...)

Optimal quantities
and related partial differential equations

Related PDE

- Value function $V(x,t)$ — HJB equation

$$V_t - \frac{1}{2}\lambda^2 \frac{V_x^2}{V_{xx}} = 0 \quad ; \quad V(x,T) = U(x)$$

- Wealth function $H(x,t)$ — heat equation

$$r\big(H(x,t),t\big) = H_x(x,t)$$

$$H_t + \frac{1}{2}\lambda^2 H_{xx} = 0 \quad ; \quad H(x,T) = I(e^{-x}) \quad , \quad I = (U')^{(-1)}$$

- Risk tolerance $r(x,t)$ — fast diffusion equation

$$r_t + \frac{1}{2}\lambda^2 r^2 r_{xx} = 0 \quad ; \quad r(x,T) = -\frac{U'(x)}{U''(x)}$$

- Risk aversion $\gamma(x,t)$ — porous medium equation

$$\gamma_t - \frac{1}{2}\lambda^2 \left(\frac{1}{\gamma}\right)_{xx} = 0 \quad ; \quad \gamma(x,T) = -\frac{U''(x)}{U'(x)}$$

Related PDE and optimal processes

- Wealth function $H(x,t)$ — heat equation

$$r\big(H(x,t),t\big) = H_x(x,t)$$

$$H_t + \frac{1}{2}\lambda^2 H_{xx} = 0 \quad ; \quad H(x,T) = I(e^{-x}) \quad , \quad I = (U')^{(-1)}$$

- Optimal wealth process (for convenience, initial time is set at zero)

$$X_t^{*,x} = H\left(H^{(-1)}(x,0) + \lambda^2 t + \lambda W_t, t\right)$$

- Optimal stock allocation process

$$\pi_t^{*,x} = \frac{\lambda}{\sigma} H_x\left(H^{(-1)}(X_t^{*,x},t),t\right) = \frac{\lambda}{\sigma} H_x\left(H^{(-1)}(x,0) + \lambda^2 t + \lambda W_t, t\right)$$

The above optimal processes, $X_t^{*,x}$ and $\pi_t^{*,x}$, are readily constructed via duality arguments but the above alternative representations are quite convenient for addressing the questions herein.

Temporal and spatial properties of optimal portfolios

Spatial monotonicity of local risk tolerance

Result: If the investor's risk tolerance $RT(x) = -\dfrac{U'(x)}{U''(x)}$ is increasing, then, for all $t \in [0, T)$, the local risk tolerance $r(x,t)$ is also increasing in x.

Proof: Recall that $r(H(x,t),t) = H_x(x,t)$ with

$$\begin{cases} H_t + \dfrac{1}{2}\lambda^2 H_{xx} = 0 \quad ; \quad H(x,T) = I(e^{-x}) \\ H_{xt} + \dfrac{1}{2}\lambda^2 H_{xxx} = 0 \quad ; \quad H_x(x,T) = -e^{-x} I'(e^{-x}) > 0 \end{cases}$$

Therefore, $\quad r_x(x,t) = \dfrac{H_{xx}(H^{(-1)}(x,t),t)}{H_x(H^{(-1)}(x,t),t)}$

Similarly, $\quad RT'(x) = \dfrac{H_{xx}(H^{(-1)}(x,T),T)}{H_x(H^{(-1)}(x,T),T)}$ and $RT'(x) > 0$

A direct application of the comparison principle for the heat equations satisfied by H_x and H_{xx} yields the result. The above provides a short proof of Borell's result.

Spatial concavity/convexity of local risk tolerance

Result: If the investor's risk tolerance $RT(x)$ is concave/convex, then, for all $t \in [0, T)$, the local risk tolerance $r(x,t)$ is also concave/convex.

Proof: Using again that $r(H(x,t),t) = H_x(x,t)$, we deduce

$$r_{xx}(x,t) = \frac{1}{r^2(x,t)} \det \begin{vmatrix} H_x(H^{(-1)},t) & H_{xx}(H^{(-1)},t) \\ H_{xx}(H^{(-1)},t) & H_{xxx}(H^{(-1)},t) \end{vmatrix}$$

Similarly

$$RT''(x) = \frac{1}{RT^2(x)} \det \begin{vmatrix} H_x(H^{(-1)},T) & H_{xx}(H^{(-1)},T) \\ H_{xx}(H^{(-1)},T) & H_{xxx}(H^{(-1)},T) \end{vmatrix}$$

The sign of the above Hankel determinants depends on the log concavity/log convexity of the function $H_x(x,t)$, $0 \le t \le T$.

Proof (con'd)

On the other hand, H_x solves the heat equation

$$H_{xt} + \frac{1}{2}\lambda^2 H_{xxx} = 0 \quad ; \quad H_x(x,T) = -e^{-x} I'(e^{-x})$$

Moreover, $RT(x)$ is concave/convex iff $H_x(x,T)$ is log concave/log convex.

Classical results for the heat equation (e.g., Keady (1990)) yield the preservation of log concavity/log convexity of the solution $H_x(x,t)$.

Temporal monotonicity of risk tolerance

Result: If the investor's risk tolerance $RT(x)$ is concave/convex, then, the local risk tolerance $r(x,t)$ is increasing/decreasing with respect to time.

Proof: The fast diffusion equation yields

$$r_t + \frac{1}{2}\lambda^2 r^2 r_{xx} = 0 \quad ; \quad r(x,T) = RT(x)$$

If $RT(x)$ is concave/convex, the previous result yields that $r(x,t)$ is also concave/convex.

Then, the above equation gives that $r_t > 0$ (< 0).

Therefore, if the investor's risk tolerance $RT(x)$ is concave/convex, then, the optimal feedback stock allocation, $\pi^*(x,t) = \frac{\lambda}{\sigma} r(x,t)$, increases/decreases as the time to maturity decreases.

Robustness of risk tolerance
and dependence on market parameters

Comparison result

Result: Assume that $RT^1(x) \leq RT^2(x)$, all $x \geq 0$. Then, for all $x \geq 0$,

$$r^1(x,t) \leq r^2(x,t) \ , \qquad t \in [0,T) \ .$$

Proof: Recall that r solves $r_t + \frac{1}{2}\lambda^2 r^2 r_{xx} = 0$.
Comparison for such equations might not hold.
Let $F(x,t) = r^2(x,t)$. Then F solves the quasilinear equation

$$F_t + \frac{1}{2} F F_{xx} - \frac{1}{4} F_x^2 = 0 \quad ; \quad F(x,t) = RT^2(x)$$

Establish comparison for the above equation (use results of Fukuda et al. (1993)).
Use positivity of risk tolerance to conclude.

Previous comparison results were produced for $RT^i(x)$ being linear ((Huang-Z.), (Back et al.)). The above result was proved by a combination of duality and penalization arguments by Xia.

Consequences of the comparison result

Recall that $\pi^*(x,t)$ and $r(x,t)$ solve

$$\pi^*_t + \frac{1}{2}\sigma^2\pi^*\pi^*_{xx} = 0 \quad ; \quad \pi^*(x,T) = \frac{\lambda}{\sigma}RT(x)$$

$$r_t + \frac{1}{2}\lambda^2 r^2 r_{xx} = 0 \quad ; \quad r(x,t) = RT(x)$$

Result: If $RT(x)$ is concave/convex, then $r(x,t)$ is increasing/decreasing with respect to the stock's Sharpe ratio λ.

Proof: $RT(x)$ concave $\longrightarrow r(x,t)$ concave. If $\lambda_1 \leq \lambda_2$, then $r_1(x,t)$ satisfies

$$r_{1,t} + \frac{1}{2}\lambda_1^2 r_1^2 r_{1,xx} = r_{1,t} + \frac{1}{2}\lambda_2^2 r_1^2 r_{1,xx} + \frac{1}{2}\underbrace{(\lambda_1^2 - \lambda_2^2) r_1^2 r_{1,xx}}_{>0} \geq r_{1,t} + \frac{1}{2}\lambda_2^2 r_1^2 r_{1,xx}\,.$$

Therefore, r_1 is a subsolution to the equation satisfied by r_2, and, thus

$$r_1(x,t) \leq r_2(x,t)$$

Consequences of the comparison result (con'd)

- If $RT(x)$ is concave/convex, then $r(x,t)$ is increasing/decreasing with respect to the mean rate of return, μ, and decreasing/increasing with respect to the volatility σ.

- The optimal portfolio $\pi^*(x,t;\sigma,\lambda) = \dfrac{\lambda}{\sigma} r(x,t;\sigma,\lambda)$ is always increasing in λ and decreasing in σ.

- If $RT(x)$ is concave/convex, then for all (x,t),

$$r(x,t) \underset{(\geq)}{\leq} RT'(0)x \qquad \text{and} \qquad \pi^*(x,t) \leq \frac{\lambda}{\sigma} RT'(0)x$$

The optimal wealth process
and space-time harmonic functions

The optimal wealth process

$$X_t^{*,x} = H\left(H^{(-1)}(x,0) + \lambda^2 t + \lambda W_t, t\right)$$

where

$$H_t + \frac{1}{2}\lambda^2 H_{xx} = 0 \quad ; \quad H(x,T) = I(e^{-x})$$

- Therefore, the process

$$H^{(-1)}(X_t^{*,x}) - H^{(-1)}(x,0) = \lambda^2 t + \lambda W_t$$

  is independent of risk preferences, across all investors!

- The function $H^{(-1)}$ plays a very important role in several key calculations.

(See, also, a recent preprint of Shkolnikov (2012))

The inverse wealth function $H^{(-1)}$

- The function $h(x,t) = H^{(-1)}(x,t)$ solves the "reciprocal" HJB equation,

$$h_t + \frac{1}{2}\lambda^2 \frac{h_{xx}}{h_x^2} = 0 \quad ; \quad h(x,T) = \Big(I(e^{-x})\Big)^{(-1)}$$

- Spatial increment

$$H^{(-1)}(y,t) - H^{(-1)}(x,t) = \int_x^y \gamma(z,t)\, dz$$

- Temporal increment

$$H^{(-1)}(x,t) - H^{(-1)}(x,s) = \frac{1}{2}\int_s^t r_x(x,\rho)\, d\rho$$

Important application

The transition probability of the optimal wealth process

$$\mathbb{P}\left(X_t^{*,x} \leq y\right) = \mathbb{P}\left(H\left(H^{(-1)}(x,0) + \lambda^2 t + \lambda W_t, t\right) \leq y\right)$$

$$= \mathbb{P}\left(\lambda^2 t + \lambda W_t \leq H^{(-1)}(y,t) - H^{(-1)}(x,0)\right)$$

$$= \mathbb{P}\left(\lambda W_t \leq \underbrace{\left(H^{(-1)}(y,t) - H^{(-1)}(x,t)\right)}_{\substack{\text{aggregate risk aversion} \\ \text{(space)}}} + \underbrace{\left(H^{(-1)}(x,t) - H^{(-1)}(x,0)\right)}_{\substack{\text{aggregate derivative} \\ \text{of risk tolerance} \\ \text{(time)}}} - \lambda^2 t\right)$$

Therefore,

$$\mathbb{P}(X_t^{*,x} \le y) = \mathbb{P}\Big(W_t \le \frac{1}{\lambda}\Big(\int_x^y \gamma(z,t)\,dz + \frac{1}{2}\int_0^t r_x(x,s)\,ds\Big) - \lambda t\Big)$$

$$= \mathcal{N}\left(\frac{1}{\lambda\sqrt{t}}A(x,y,0,t) - \lambda\sqrt{t}\right) \quad ;$$

$$A(x,y,0,t) = \int_x^y \gamma(z,t)\,dz + \frac{1}{2}\int_0^t r_x(x,s)\,ds$$

Moreover,

- $$\frac{\partial}{\partial y}\mathbb{P}(X_t^{*,x} \le y) = \frac{1}{\lambda\sqrt{t}}\gamma(y,t)n\Big(\frac{1}{\lambda\sqrt{t}}A(x,y,0,t) - \lambda\sqrt{t}\Big)$$

- $$\frac{\partial}{\partial x}\mathbb{P}(X_t^{*,x} \le y) = \left(-\frac{\gamma(x,t)}{\lambda\sqrt{t}} + \frac{1}{2}\int_0^t r_{xx}(x,s)\,ds\right) n\Big(\frac{1}{\lambda\sqrt{t}}A(x,y,0,t) - \lambda\sqrt{t}\Big)$$

- $$\frac{\partial}{\partial t}\mathbb{P}(X_t^{*,x} \le y) = \frac{\partial}{\partial t}\left(\frac{1}{\lambda\sqrt{t}}A(x,y,0,t) - \lambda\sqrt{t}\right) n\left(\frac{1}{\lambda\sqrt{t}}A(x,y,0,t) - \lambda\sqrt{t}\right)$$

Special case: $y = x$

$$\mathbb{P}(X_t^{*,x} \le x) = \mathbb{P}\left(W_t \le \frac{1}{2\lambda\sqrt{t}} \int_0^t r_x(x,s)\,ds - \lambda\sqrt{t}\right)$$

- $\dfrac{\partial \mathbb{P}}{\partial x}(X_t^{*,x} \le x) = \left(\dfrac{1}{2\lambda\sqrt{t}} \displaystyle\int_0^t r_{xx}(x,s)\,ds\right) n\left(\dfrac{1}{2\lambda\sqrt{t}} \displaystyle\int_0^t r_x(x,s)\,ds - \lambda\sqrt{t}\right)$
- $\dfrac{\partial \mathbb{P}}{\partial t}(X_t^{*,x} \le x) = \dfrac{\lambda}{2\sqrt{t}} \left(\dfrac{1}{\lambda^2} r_x\,(x,t) - \dfrac{1}{2\lambda^2 t} \displaystyle\int_0^t r_x\,(x,s)\,ds - 1\right) n\left(\dfrac{1}{2\lambda\sqrt{t}} \displaystyle\int_0^t r_x(x,s)\,ds - \lambda\sqrt{t}\right)$

Therefore,

- If $RT(x)$ is concave/convex, then $\mathbb{P}(X_t^{*,x} \le x)$ is decreasing/increasing with respect to x, for all $t \in [0,T)$
- If $RT'(x) < \lambda^2$, then $\mathbb{P}(X_t^{*,x} \le x)$ is decreasing with respect to t, for all $x \ge 0$; stricter bounds may be obtained from further assumptions on $RT'(x)$.

Extensions

Temporal propagation of key properties at maturity

Some properties at T which also hold at $t \in [0, T)$

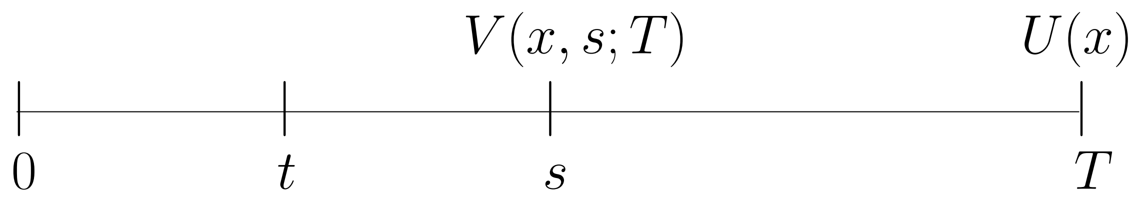
- Monotonicity of utility function
- Concavity of utility function
- Monotonicity of absolute risk tolerance
- Monotonicity of relative risk tolerance
- Concavity/convexity of absolute risk tolerance
- Positivity of prudence

Are there other meaningful and intuitive properties (qualitative or structural) which also propagate?

Investment horizon flexibility

Investment horizon flexibility

- So far,

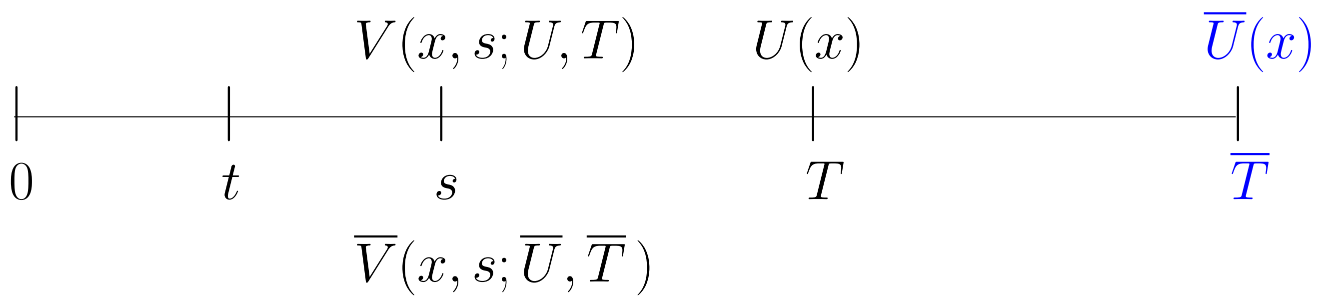


- What if the investor decides at intermediate time, say $s \in (t, T)$, to prolong the investment horizon?

- Can this be done? How and how far out? What criterion do we impose in the "new horizon"?

Flexible investment horizon, optimality and time consistency

Essentially, we are looking for $\overline{T}$ and $U_{\overline{T}}$ such that



We must have

$$V(x, s; U, T) = \overline{V}(x, s; \overline{U}, \overline{T}) \ !$$

Is this always possible?

Main results

- Let $I(x) = (U')^{(-1)}(x)$. Then, if the function $I(e^{-x})$ is absolutely monotonic, the Merton problem can be extended for every $\overline{T} > T$.

- If $I(e^{-x})$ is absolutely monotonic, the Bernstein-Widder theorem yields, for a positive finite measure ν,

$$I(e^{-x}) = \int_0^{+\infty} e^{xy} \nu(dy)$$

- Therefore, $I(x)$ is completely monotonic of the form,

$$I(x) = \int_0^{+\infty} x^{-y} \nu(dy)$$

- Moreover, if $I(x) = (U')^{(-1)}(x)$ is of this form, the inverse marginal value function is of the same form, i.e.

$$V_x^{(-1)}(x,t) = \int_0^{+\infty} x^{-y} \nu(t, dy) \qquad 0 < t < T$$

- In other words, complete monotonicity of $(U')^{(-1)}(x)$ at T is inherited to the inverse of the marginal value function.

- This is in contrast of classical results of complete monotonicity of U' (Brockett-Golden, Hammond, Gaballé and Pomansky, Bennett,...)

Summary of results

$$\begin{array}{ccc}
U'(x) \text{ C.M.} & & I(x) \text{ C.M.} \\
\downarrow & & \downarrow \\
U(x) \text{ stochastic dominance of any degree} & & U(x) \text{ stochastic dominance up to degree 3} \\
\not\downarrow & & \downarrow \\
V_x(x,t) \text{ fails to inherit C.M.} & & V_x^{(-1)}(x,t) \text{ preserves C.M.} \\
 & & \downarrow \\
 & & \text{Merton problem extends to arbitrary horizon}
\end{array}$$

$$\overset{?}{\underset{\text{No}}{\longrightarrow\;\longleftarrow}}$$

U' C.M. $\not\Longrightarrow I$ C.M.