Viscosity Solutions of Fully Nonlinear Path Dependent PDEs

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Happy Birthday Ioannis

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Outline

Parabolic nonlinear path-dependent PDEs

Let $\Omega = \{\omega \in C^0([0,T], \mathbb{R}^d), \omega_0 = 0\}$, B canonical process, Φ the corresponding filtration

Our objective : wellposedness theory for the equation :

$$\Big\{ -\partial_t u - F(., u, \partial_\omega u, \partial_{\omega\omega} u) \Big\}(t,\omega) = 0 \quad \text{for} \quad t < T,\ \omega \in \Omega$$
$$u(T,\omega) = g(\omega)$$

where $g(\omega) = g\big((\omega_s)_{s\le T}\big)$ and $F(t,\omega,y,z,\gamma)$ is $\mathbb{F}-$prog. meas. map :

$$F : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \longrightarrow \mathbb{R}$$

and the unknown process $u(t,\omega)$ is prog. meas.

Time and space derivatives

• Time derivative introduced by Dupire :

$$\partial_t u(t,\omega) := \lim_{h \searrow 0} \frac{u(t+h, \omega_{t\wedge .}) - u(t,\omega)}{h} \quad \text{if exists}$$

• Space derivatives : $u(t,\omega) \in C^{1,2}$ if there exist continuous process, denoted $\partial_\omega u$, $\partial_{\omega\omega} u$, such that Itô's formula holds :

$$du \quad = \quad \partial_t u dt + \frac{1}{2}\partial_{\omega\omega} u d\langle B\rangle + \partial_\omega dB \quad (\ldots)$$

Remark If $\partial_t u, \partial_\omega u, \partial_{\omega\omega} u$ in Dupire sense exist and continuous bounded, then Itô's formula holds true

Example : Backward SDE

Find $\mathbb{F}-$progressively measurable processes (Y, Z) such that :

$$Y_t = g - \int_t^T f(s, \omega, Y_s, Z_s)ds + \int_t^T Z_s dB_s$$

Pardoux and Peng 1990 : f Lipschitz, $\mathbb{E}\int_0^T |f_t(0,0)|^2 dt < \infty$, the is a unique solution (Y, Z) in the space

$$\|Y\|_{\mathbb{S}^2} := \mathbb{E}\sup_{t\le T}|Y_t|^2 < \infty \quad \text{and} \quad \|Y\|_{\mathbb{H}^2} := \mathbb{E}\int_t^T |Z_t|^2 dt < \infty$$

If $Y(t,\omega) \in C^{1,2}$, then $Z_t = \partial_\omega Y_t$ and

$$\partial_t Y + \frac{1}{2}\partial_{\omega\omega} Y = -f(., Y, \partial_\omega Y) \quad \text{and} \quad Y(T,.) = g$$

Example 2 : Second order BSDE

• $\mathbb{P} \in \mathcal{P} := \{\mathbb{P}^\sigma : \underline{\sigma} \le \sigma \le \overline{\sigma}\}$: $dW_t^\sigma := \sigma_t^{-1} dB_t$ is a $\mathbb{P}^\sigma$-Brownian motion and W^σ and B induce the same $\mathbb{P}^\sigma$-augmented filtration.

• Second order BSDE (Cheridito-Soner-T.-Victoir 07, Soner-T.-Zhang 11) :

$$dY_t = -f(t,\omega,Y_t,Z_t,\sigma_t^2)dt + Z_t dB_t - dK_t, \quad Y_T = g(\omega),$$
$$\mathbb{P}^\sigma\text{-a.s. for all } \sigma.$$

◇ When $f = 0$, Y is a G-martingale (Peng 06)

• If $Y(t,\omega) \in C^{1,2}$, then $Z_t = \partial_\omega Y_t$, and

$$\partial_t Y + F(t,\omega,Y,\partial_\omega Y,\partial_{\omega\omega} Y) = 0, \quad Y(T,.) = g$$
$$F(t,\omega,y,z,\gamma) := \sup_{\underline{\sigma} \le \sigma \le \overline{\sigma}} \left\{ \frac{1}{2}\sigma^2\gamma + f(t,\omega,y,z,\sigma^2) \right\}.$$

A larger class of fully nonlinear PPDEs

• Dynamic programming equation for a non-Markov stochastic control problem

• Dynamic programming equations for non-Markov differential games

Not accessible from the existing 2BSDE results

Example 3 : Backward Stochastic PDE

- BSPDE :

$$du(t,x,\omega) = -f(t,x,\omega,u,\partial_x u,\partial_{xx}u,\beta,\partial_x\beta)dt + \beta(t,x,\omega)dB_t$$

- Functional Itô's formula :

$$du(t,x,\omega) = (\partial_t u + \frac{1}{2}\partial_{\omega\omega}u)dt + \partial_\omega u dB_t$$

Then $\beta = \partial_\omega u$, $\beta_x = \partial_{x\omega}u$, and we arrive at the PPDE :

$$\partial_t u + \frac{1}{2}\partial_{\omega\omega}u + f(t,x,\omega,u,\partial_x u,\partial_{xx}u,\partial_\omega u,\partial_{x\omega}u) = 0.$$

Our approach allows to handle General mixed PPDE :

$$\partial_t u + F(t,x,\omega,u,\partial_x u,\partial_\omega u,\partial_{xx}u,\partial_{x\omega}u,\partial_{\omega\omega}u) = 0$$
$$u(T,x,\omega) = g(x,\omega).$$

Applications of BSPDEs

- Solving non-Markov FBSDEs by the method of decoupling field (Zhang 06, Ma-Yin-Zhang 10, Ma-Wu-Zhang-Zhang 10)

- Control of Stochastic PDEs

- Rate function for a large deviation problem (Ma-T.-Zhang)

$$\partial_t u - \frac{1}{2}\left|\partial_\omega u + \partial_x u\sigma(t,x,\omega)\right|^2 = 0, u(T,x,\omega) = g(x,\omega)$$

(Path-dependent Eikonal equation)

Why viscosity solutions of PPDEs

We want to adapt the theory of viscosity solutions to the present case

- To obtain wellposedness for a larger class of equations
- Powerful stability result
- Easy !

Main difficulty : the paths space Ω is not locally compact

Recall standard viscosity solutions

$f(x, y, z, \gamma)$ nondecreasing in γ. Consider the PDE :

$$\text{(E)} \qquad \big\{ -f(., v, Dv, D^2v) \big\}(x) = 0, \quad x \in \mathcal{O} \quad \text{(open subset of } \mathbb{R}^d\text{)}$$

Exercise For $v \in C^2(\mathcal{O})$, the following are equivalent :

(i) v is a supersolution of (E)

(ii) For all $(x_0, \phi) \in \mathcal{O} \times C^2(\mathcal{O})$:

$$(\phi - v)(x_0) = \max_{\mathcal{O}}(\phi - v) \quad \Longrightarrow \quad \big\{ -f(., v, D\phi, D^2\phi) \big\}(x_0) \geq 0$$

Intuition from consistency with classical solutions (1)

Since $v(t,x)$ is a classical supersolution :

$$\begin{aligned}
0 &\le \big\{-\partial_t v - F(.,v,Dv,D^2v)\big\}(t_0,x_0)\\
&= \big\{-\partial_t \phi - F(.,v,D\phi,D^2\phi)\big\}(t_0,x_0)\\
&\quad + \Big(\underbrace{\partial_t(\phi-v)}_{=0} + F_z(\ldots)\underbrace{D(\phi-v)}_{=0} + F_\gamma(\ldots)\underbrace{D^2(\phi-v)}_{\le 0}\Big)(t_0,x_0)\\
&\le \big\{-\partial_t \phi - F(.,v,D\phi,D^2\phi)\big\}(t_0,x_0)
\end{aligned}$$

by $F_\gamma \ge 0$, the first and second order conditions for $(\phi - v)(x_0) = \max_{\mathcal{O}}(\phi - v)$

Intuition from consistency with classical solutions (2)

Since $u(t,\omega)$ is a classical supersolution :

$$\begin{aligned}
0 &\le \big\{-\partial_t u - F(.,u,\partial_\omega u,\partial_{\omega\omega}u)\big\}(t_0,\omega_0)\\
&= \big\{-\partial_t\varphi - F(.,u,\partial_\omega\varphi,\partial_{\omega\omega}\varphi)\big\}(t_0,\omega_0)\\
&\quad+\underbrace{\big(\partial_t(\varphi-u)+F_z(\dots)\partial_\omega(\varphi-u)+F_\gamma(\dots)\partial_{\omega\omega}(\varphi-u)\big)(t_0,\omega_0)}_{=:R(t_0,\omega_0)\le 0 \quad \text{Needed}}\\
&\le \big\{-\partial_t\varphi - F(.,u,\partial_\omega\varphi,\partial_{\omega\omega}\varphi)\big\}(t_0,\omega_0)
\end{aligned}$$

Remark $d(\varphi-u)(t_0,\omega_0) = R(t_0,\omega_0)dt + \partial_\omega(\varphi-u)(t_0,\omega_0)dB$, $\hat{\mathbb{P}}-$a.s.

where $\hat{\mathbb{P}}$ is the probability measure on Ω under which

$$dB_t = \hat\alpha_t dt + \hat\beta_t d\hat W_t, \quad \hat\alpha := F_p(\dots), \quad \hat\beta := \sqrt{2F_\gamma(\dots)}$$

$\hat W$ is a $\hat{\mathbb{P}}-$Brownian motion

Intuition from consistency with classical solutions (2)

Since $u(t,\omega)$ is a classical supersolution :

$$
\begin{aligned}
0 &\le \Big\{ -\partial_t u - F(.,u,\partial_\omega u,\partial_{\omega\omega} u)\Big\}(t_0,\omega_0)\\
&= \Big\{ -\partial_t \varphi - F(.,u,\partial_\omega \varphi,\partial_{\omega\omega} \varphi)\Big\}(t_0,\omega_0)\\
&\quad + \underbrace{\Big(\partial_t(\varphi-u) + F_z(\ldots)\partial_\omega(\varphi-u) + F_\gamma(\ldots)\partial_{\omega\omega}(\varphi-u)\Big)(t_0,\omega_0)}_{=:R(t_0,\omega_0)\le 0\quad \text{Needed}}\\
&\le \Big\{ -\partial_t \varphi - F(.,u,\partial_\omega \varphi,\partial_{\omega\omega} \varphi)\Big\}(t_0,\omega_0)
\end{aligned}
$$

Remark $d(\varphi-u)(t_0,\omega_0) = R(t_0,\omega_0)dt + \partial_\omega(\varphi-u)(t_0,\omega_0)dB$, $\hat{\mathbb{P}}-$a.s.

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$$

$\hat W$ is a $\hat{\mathbb{P}}-$Brownian motion

Intuition from consistency with classical solutions (3)

So we want

$$d(\varphi - u)(t_0, \omega_0) = \underbrace{R(t_0, \omega_0)}_{\leq 0} dt + \partial_\omega(\varphi - u)(t_0, \omega_0) dB, \quad \hat{\mathbb{P}} - \text{a.s.}$$

i.e. $\varphi - u$ $\hat{\mathbb{P}}-$supermartingale locally to the right of (t_0, ω_0)

- No control on $\hat{\mathbb{P}}$, so assuming F is L_0-Lipschitz

$$\text{Choose } \varphi \text{ s.t. } \varphi - u \ \ \mathbb{P}^{\alpha,\beta} - \text{supermart. locally } (t_0+, \omega_0)$$
$$\text{for all } |\alpha|, |\tfrac{1}{2}\beta^2| \leq L_0$$

Hence, for some stopping time $\mathrm{h} > t_0$:

$$(\varphi - u)(t_0, \omega) = \sup_{\tau \text{ stop.}} \sup_{\alpha,\beta} \mathbb{E}^{\mathbb{P}^{\alpha,\beta}}\left[(\varphi - u)_{\tau \wedge \mathrm{h}}\right]$$

Outline

Canonical space and continuity of random fields

$\Omega := \{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = 0\}$ and $\Lambda := [0,T] \times \Omega$

B : canonical process, $\mathbb{F}$ the corresponding filtration

$\|\omega\|_t := \sup_{0 \le s \le t} |\omega_s|$

Definiton An $\mathbb{F}$-prog. meas. process $u : \Lambda \to \mathbb{R}$ is in $USC(\Lambda)$ if u is right continuous in t, and there exists a modulus of continuity function ρ s.t.

$$u(t,\omega) - u(t',\omega') \;\le\; \rho\big(t' - t + \|\omega_{t\wedge\cdot} - \omega'_{t'\wedge\cdot}\|_T\big) \quad \text{whenever} \quad t \le t'$$

$u \in LSC(\Lambda)$ if $-u \in USC(\Lambda)$

Capacity and nonlinear expectations

- $L > 0$, $\mathcal{P}_L$: set of prob. meas. $\mathbb{P}$ on Ω s.t.

$$|\alpha^{\mathbb{P}}| \le L, \quad 0 \le \beta^{\mathbb{P}} \le \sqrt{2L}\, I_d, \quad dB_t = \beta_t^{\mathbb{P}} dW_t^{\mathbb{P}} + \alpha_t^{\mathbb{P}} dt, \quad \mathbb{P}\text{-a.s.}$$

for some $\mathbb{F}-$prog. meas. processes $\alpha^{\mathbb{P}}, \beta^{\mathbb{P}}$, and some d-dimensional $\mathbb{P}$-Brownian motion $W^{\mathbb{P}}$

- $\mathcal{P}_\infty := \bigcup_{L>0} \mathcal{P}_L$

- For $\xi \in \mathbb{L}^1(\mathcal{F}_T, \mathcal{P}_L)$, define the nonlinear expectation :

$$\overline{\mathcal{E}}^L[\xi] = \sup_{\mathbb{P}\in\mathcal{P}_L} \mathbb{E}^{\mathbb{P}}[\xi] \quad \text{and} \quad \underline{\mathcal{E}}^L[\xi] = \inf_{\mathbb{P}\in\mathcal{P}_L} \mathbb{E}^{\mathbb{P}}[\xi] = -\overline{\mathcal{E}}^L[-\xi]$$

Differentiability of processes

- For $u \in C^0(\Lambda)$, the right time-derivative is defined by Dupire :

$$\partial_t u(t,\omega) := \lim_{h\to 0, h>0} \frac{1}{h}\Big[u\big(t+h,\omega_{\cdot\wedge t}\big) - u\big(t,\omega\big)\Big], \quad t<T$$
$$\partial_t u(T,\omega) := \lim_{t<T, t\uparrow T} \partial_t u(t,\omega)$$

whenever the limits exist

- $u \in C^{1,2}(\Lambda)$ if $u \in C^0(\Lambda)$, $\partial_t u \in C^0(\Lambda)$, and there exist $\partial_\omega u \in C^0(\Lambda, \mathbb{R}^d)$, $\partial_{\omega\omega} u \in C^0(\Lambda, \mathbb{S}^d)$ such that for all $\mathbb{P} \in \mathcal{P}_\infty$:

$$du_t = \partial_t u_t dt + \partial_\omega u_t \cdot dB_t + \frac{1}{2}\partial_{\omega\omega} u_t : d\left\langle B \right\rangle_t, \quad \mathbb{P}\text{-a.s.}$$

$\partial_\omega u$ and $\partial_{\omega\omega} u$, if exist, are unique

Optimal stopping under nonlinear expectation

- For X bounded prog. meas. define for $(t,\omega)\in\Lambda$:

$$\overline{\mathcal{S}}^L_t[X](\omega) := \sup_{\tau\in\mathcal{T}^t} \overline{\mathcal{E}}^L_t\left[X^{t,\omega}_\tau\right], \quad \text{and} \quad \underline{\mathcal{S}}^L_t[X](\omega) := -\overline{\mathcal{S}}^L_t[-X](\omega)$$

Theorem

Let $X\in USC_b(\Lambda)$, $\mathrm{h}\in\mathcal{H}$, and define

$$Y := \overline{\mathcal{S}}^L[X_{\cdot\wedge\mathrm{h}}] \quad \textit{and} \quad \tau^* := \mathrm{h}\wedge\inf\{t : Y_t = X_t\}$$

Then $Y_{\tau^*} = X_{\tau^*}$, Y is an $\overline{\mathcal{E}}^L-$supermartingale, and an $\overline{\mathcal{E}}^L$-martingale on $[0,\tau^*]$.

Outline

Nonlinearity

Given a generator $F : \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$, consider :

$$\mathcal{L}u(t,\omega) := -\partial_t u(t,\omega) - F(t,\omega,u(t,\omega),\partial_\omega u(t,\omega),\partial_{\omega\omega} u(t,\omega))$$

Want to solve :

$$\mathcal{L}u(t,\omega) \;=\; 0, \quad 0 \le t < T, \; \omega \in \Omega$$

Main assumptions

Assumption F1 $F(t, \omega, y, z, \gamma)$ nondecreasing in γ and satisfies :

(i) $F(\cdot, y, z, \gamma)$ is $\mathbb{F}$-prog. meas., and $\|F(\cdot, 0, 0, 0)\|_\infty < \infty$.

(ii) F is uniformly continuous in ω

(iii) F is uniformly Lipschitz in (y, z, γ)

Smooth test processes

$$\underline{\mathcal{A}}^L u(t,\omega) := \left\{ \varphi \in C^{1,2}(\Lambda^t) : \exists\, \mathrm{h} \in \mathcal{H}^t,\ (\varphi - u^{t,\omega})_t(\mathbf{0}) = \underline{\mathcal{S}}_t^L\left[(\varphi - u^{t,\omega})_{.\wedge \mathrm{h}}\right] \right\}$$

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Definition

Definition $u \in USC_b(\Lambda)$ (resp. $LSC_b(\Lambda)$) :

• u viscosity L-subsolution (resp. L-supersolution) of PPDE if :

$$-\partial_t \varphi_t(0) - F\big(t, \omega, u(t,\omega), \partial_\omega \varphi_t(0), \partial_{\omega\omega}\varphi_t(0)\big) \le \text{(resp. } \ge \text{) } 0$$

for all $(t,\omega) \in [0,T) \times \Omega$ and $\varphi \in \underline{\mathcal{A}}^L u(t,\omega)$ (resp. $\varphi \in \overline{\mathcal{A}}^L u(t,\omega)$)

• u viscosity subsolution (resp. supersolution) of PPDE if $\exists\, L > 0$ s.t. u is viscosity L-subsolution (resp. L-supersolution) of PPDE

• u is a viscosity solution of PPDE if it is both a viscosity subsolution and a viscosity supersolution

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Remarks

• In the definition of $\underline{\mathcal{A}}u(t,\omega)$ and $\overline{\mathcal{A}}u(t,\omega)$, we may restrict attention the those φ such that $(\varphi - u^{t,\omega})_t(0) = 0$

• "min" and "max" can be further localized in the definition of $\underline{\mathcal{A}}u(t,\omega)$ and $\overline{\mathcal{A}}u(t,\omega)$, i.e. we may use

$$H_\varepsilon \quad := \quad H \wedge (t+\varepsilon) \wedge \inf\{s > t : |B_s^t| \geq \varepsilon\}$$

• "min" and "max" in the definition of $\underline{\mathcal{A}}u(t,\omega)$ and $\overline{\mathcal{A}}u(t,\omega)$ can be taken to be strict

• Change of variable

Consistency with classical solutions

Theorem

Let Assumption F1 hold and $u \in C_b^{1,2}(\Lambda)$. Then the following assertions are equivalent :

- *u classical solution (resp. subsolution, supersolution) of PPDE*
- *u viscosity solution (resp. subsolution, supersolution) of PPDE*

Stability

Theorem

Let $(F^\varepsilon, \varepsilon > 0)$ be a family of coefficients

- *satisfying Assumptions F uniformly,*
- $F^\varepsilon \longrightarrow F$ *as* $\varepsilon \to 0$.

For fixed $L > 0$, let $(u^\varepsilon)_{\varepsilon>0}$ be such that

- u^ε *is viscosity $L-$subsolution (resp. $L-$supersolution) of PPDE with coefficients F^ε, for all $\varepsilon > 0$,*
- $u^\varepsilon \longrightarrow u$, *uniformly in Λ.*

Then u is a viscosity $L-$subsolution (resp. supersolution) of PPDE with coefficient F.

Partial comparison

Theorem

Let Assumption F1 hold. Let

- u^1 *be a bounded viscosity subsolution of PPDE,*
- u^2 *a bounded viscosity supersolution of PPDE,*
- $u^1(T,\cdot) \le u^2(T,\cdot)$.

Assume further that either u^1 or u^2 is in $\bar{C}^{1,2}(\Lambda)$. Then

$$u^1 \le u^2 \quad \text{on} \quad \Lambda$$

Outline

Terminal condition

Assumption G g is bounded and uniformly continuous in ω

Freezing ω in the generator

- Define the deterministic function on $[t,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d$:

$$f^{t,\omega}(s,y,z,\gamma) \quad := \quad F(s\wedge T,\omega_{\cdot\wedge t},y,z,\gamma)$$

- Consider the standard PDE :

$$\mathbf{L}^{t,\omega}v \quad := \quad -\partial_t v - f^{t,\omega}(s,v,Dv,D^2v) \;=\; 0, \quad (t,x)\in O_t^{\varepsilon,\eta}$$

where

$$O_t^{\varepsilon,\eta} \quad := \quad [t,(1+\eta)T)\times\{x\in\mathbb{R}^d : |x|<\varepsilon\}, \quad \varepsilon>0,\eta\geq 0$$

Additional Assumption on the generator

Assumption F2 For any small $\varepsilon > 0, \eta \geq 0$ and any $(t, \omega) \in \Lambda$, PDE is wellposed in the following sense :

(i) Comparison principle : for any viscosity subsolution $v^1 \in C^0(\bar{O}_t^{\varepsilon,\eta})$ and viscosity supersolution $v^2 \in C^0(\bar{O}_t^{\varepsilon,\eta})$, if $v^1 \leq v^2$ on $\partial O_t^{\varepsilon,\eta}$, then $v^1 \leq v^2$ in $O_t^{\varepsilon,\eta}$.

(ii) Peron's approach : given a continuous function $h : \partial O_t^{\varepsilon,\eta} \to \mathbb{R}$, the PDE with boundary condition h has a unique viscosity solution v and it satisfies $v = \overline{v} = \underline{v}$, where

$$\overline{v}(s,x) := \inf\left\{\phi(s,x) : \phi \in C^{1,2}(\bar{O}_t^{\varepsilon,\eta}),\ \mathbf{L}^{t,\omega}\phi \geq 0 \text{ in } O_t^{\varepsilon,\eta},\ \phi \geq h \text{ on } \partial O_t^{\varepsilon,\eta}\right\}$$

$$\underline{v}(s,x) := \sup\left\{\psi(s,x) : \phi \in C^{1,2}(\bar{O}_t^{\varepsilon,\eta}),\ \mathbf{L}^{t,\omega}\psi \leq 0 \text{ in } O_t^{\varepsilon,\eta},\ \phi \leq h \text{ on } \partial O_t^{\varepsilon,\eta}\right\}$$

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The main results

Theorem (Comparison)

Under Assumptions F1, F2 and G, let u^1 and u^2 be such that :

- u^1 *is a bounded viscosity subsolution of PPDE*
- u^2 *is a bounded viscosity supersolution of PPDE*
- $u^1(T,\cdot) \le g \le u^2(T,\cdot)$

Then $u^1 \le u^2$ on Λ.

Theorem (Existence)

Under Assumptions F1, F2 and G, the PPDE with terminal condition g admits a unique bounded viscosity solution $u \in C^0(\Lambda)$.

The main results

Theorem (Comparison)

Under Assumptions F1, F2 and G, let u^1 and u^2 be such that :

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Theorem (Existence)

Under Assumptions F1, F2 and G, the PPDE with terminal condition g admits a unique bounded viscosity solution $u \in C^0(\Lambda)$.

Strategy of proof (1)

Follow Peron's approach defining

$$
\begin{aligned}
\overline{u}(t,\omega) &:= \inf\left\{\varphi(t,\mathbf{0}) : \varphi \in \overline{\mathcal{D}}(t,\omega)\right\}, \\
\underline{u}(t,\omega) &:= \sup\left\{\varphi(t,\mathbf{0}) : \varphi \in \underline{\mathcal{D}}(t,\omega)\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{\mathcal{D}}(t,\omega) &:= \left\{\varphi \in \bar{C}^{1,2}(\Lambda^t) : \ (\mathcal{L}\varphi)_s^{t,\omega} \geq 0, \ s \in [t,\, T] \text{ and } \varphi_T \geq g^{t,\omega}\right\} \\
\underline{\mathcal{D}}(t,\omega) &:= \left\{\varphi \in \bar{C}^{1,2}(\Lambda^t) : \ (\mathcal{L}\varphi)_s^{t,\omega} \leq 0, \ s \in [t,\, T] \text{ and } \varphi_T \leq g^{t,\omega}\right\}
\end{aligned}
$$

Strategy of proof (2)

Proposition 1 $\overline{u}$ (resp. $\underline{u}$) is a viscosity L_0-supersolution (resp. L_0-subsolution) of PPDE with terminal condition g.

Proposition 2 $\overline{u} = \underline{u}$

Proof of wellposedness

- Propositions 1 and 2 imply that $\underline{u} = \overline{u}$ is a viscosity solution of PPDE with terminal condition g.
- By partial comparison, we have $u^1 \leq \overline{u}$ and $\underline{u} \leq u^2$. Then Proposition 2 implies $u^1 \leq u^2$.