# Stochastic Perron's Method in Linear and Nonlinear Problems

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# Happy Birthday Yannis!

Γιάννη, Χρόνια σου πολλά!

# Outline

# Summary

New look at an old (set of) problem(s).

**Disclaimer:**

- not trying to "reinvent the wheel" but provide a different view (and a new tool)

**Questions:**

- why a new look?
- how/ the tool we propose

# Stochastic Control Problems

State equation

$$\begin{cases} dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dW_t \\ X_s = x. \end{cases}$$

$X \in \mathbb{R}^n, W \in \mathbb{R}^d$

Cost functional $J(s, x, \alpha) = \mathbb{E}[\int_s^T R(t, X_t, \alpha_t)dt + g(X_T)]$

Value function

$$v(s, x) = \sup_\alpha J(s, x, \alpha).$$

Comments: all formal, no filtration, admissibility, etc. Also, we have in mind other classes of control problems as well.

# (My understanding of) Continuous-time DP and HJB's

Two possible approaches

1. analytic (direct)
2. probabilistic (study the properties of the value function)

# The Analytic approach

1. write down the DPE/HJB

$$\begin{cases} u_t + \sup_\alpha \left\{ L_t^\alpha u + R(t,x,\alpha) \right\} = 0 \\ u(T,x) = g(x) \end{cases}$$

2. solve it i.e.
   - prove existence of a smooth solution $u$
   - (if lucky) find a closed form solution $u$
3. go over verification arguments
   - proving existence of a solution to the closed-loop SDE
   - use Itô's lemma and uniform integrability, to conclude $u = v$ and the solution of the closed-loop eq. is optimal

# Analytic approach cont'd

Conclusions: the existence of a smooth solution of the HJB (with some properties) implies

1. $u = v$ (uniqueness of the smooth solution)
2. (DPP)

$$v(s,x) = \sup_{\alpha} \mathbb{E}[\int_s^{\tau} R(t, X_t, \alpha_t)dt + v(\tau, X_\tau)]$$

3. $\alpha(t,x) = \arg\max$ is the optimal feedback

Complete description: Fleming and Rishel

$\boxed{\text{smooth sol of (DPE)} \rightarrow \text{(DPP)+value fct is the unique sol}}$

# Probabilistic/Viscosity Approach

1. prove the (DPP)
2. show that (DPP) $\longrightarrow$ $v$ is a viscosity solution
3. IF viscosity comparison holds, then $v$ is the unique viscosity solution

$$\boxed{\text{(DPP)+visc. comparison} \rightarrow v \text{ is the unique visc sol(DPE)}}$$

**Meta-Theorem** If the value function is the unique viscosity solution, then finite difference schemes approximate the value function and the optimal feedback control (approximate backward induction works).

# Comments on probabilistic approach

1. quite hard (actually very hard compared to deterministic case)
   1.1 by approx with discrete-time or smooth problems (Krylov)
   1.2 work directly on the value function (El Karoui, Borkar, Hausmann, Bouchard-Touzi for a weak version)
2. non-trivial, but easier than 1: Fleming-Soner, Bouchard-Touzi
3. has to be proved separately (analytically) anyway

# Probabilistic/Viscosity Approach pushed further

Sometimes we are lucky:

- using the specific structure of the HJB can prove that a viscosity solution of the DPE is actually smooth!
- if that works we can just come back to the Analytic approach and go over step 3, i.e. we can perform verification using the smooth solution $v$ (the value function) to obtain
  1. the (DPP)
  2. Optimal feedback control $\alpha(t, x)$

$$\boxed{\text{(DPP)} \to v \text{ is visc. sol} \to v \text{ is smooth sol} \to \text{(DPP)} + \text{opt. controls}}$$

Examples: Shreve and Soner, Pham

# Viscosity solution is smooth, cont'd

- the first step is hardest to prove
- the program seems circular

**Question:** can we just avoid the first step, proving the (DPP)?
**Answer:** yes, we can use (Ishii's version of) Perron's method to construct (quite easily) a viscosity solution.

Lucky case, revisited

$$\boxed{\text{Perron} \rightarrow \text{visc. sol} \rightarrow \text{smooth sol} \rightarrow \text{unique}+(\text{DPP}) +\text{opt. controls}}$$

Example: Janeček, S.

**Comments:**

- old news for PDE
- the new approach is analytic/direct

# Perron's method

**General Statement:** sup over sub-solutions and inf over super-solutions are solutions.

$$v^- = \sup_{w \in \mathscr{U}^-} w, v^+ = \inf_{w \in \mathscr{U}^+} w \textit{ are solutions}$$

**Ishii's version of Perron (1984):** sup over viscosity sub-solutions and inf over viscosity super-solutions are viscosity solutions.

$$v^- = \sup_{w \in \mathscr{U}^{-,visc}} w, v^+ = \inf_{w \in \mathscr{U}^{+,visc}} w \textit{ are viscosity solutions}$$

**Question:** why not inf/sup over classical super/sub-solutions?
**Answer:** Because one cannot prove (in general/directly) the result is a viscosity solution. The classical solutions are not enough (the set of classical solutions is not stable under max or min).

# Objective

Provide a method/tool to replace the first two steps in the program

$$\boxed{\text{Perron} \rightarrow \text{visc. sol} \rightarrow \text{smooth sol} \rightarrow \text{unique+(DPP) +opt. controls}}$$

in case one cannot prove viscosity solutions are smooth ("the unlucky case")

$$\boxed{\text{New method/tool} \rightarrow \textbf{construct} \text{ a visc. sol } u \rightarrow u = v \text{ +(DPP)}}$$

Why not try a version of Perron's method?

# Perron's method, recall

(Ishii's version) Provides viscosity solutions of the HJB

$$v^- = \sup_{w \in \mathscr{U}^{-,visc}} w, v^+ = \inf_{w \in \mathscr{U}^{+,visc}} w$$

**Problem:**

- $w$ does NOT compare to the value function $v$ UNLESS one proves $v$ is a viscosity solutions already AND the viscosity comparison
- if we ask $w$ to be classical semi-solutions, we cannot prove that the inf/sup are viscosity solutions

# Main Idea

Perform Perron's Method over a class of semi-solutions which are

- weak enough to conclude (in general/directly) that $v^-, v^+$ are viscosity solutions
- strong enough to compare with the value function **without studying the properties of the value function**

We know that

$\boxed{\text{classical sol} \rightarrow \text{(DPP)} \rightarrow \text{viscosity sol}}$

Actually, we have

$\boxed{\text{classical semi-sol} \rightarrow \text{half-(DPP)} \rightarrow \text{viscosity semi-sol}}$

Translation
"half (DPP)= stochastic semi-solution"
**Main property:** stochastic sub and super-solutions DO compare with the value function $v$!

# Stochastic Perron Method, quick summary

**General Statement:**

- supremum over stochastic sub-solutions is a viscosity (super)-solution

$$v_* = \sup_{w \in \mathscr{U}^{-,stoch}} w \le v$$

- infimum over stochastic super-solutions is a viscosity (sub)-solution

$$v^* = \inf_{w \in \mathscr{U}^{+,stoch}} w \ge v$$

Conclusion:

$$v_* \le v \le v^*$$

IF we have a viscosity comparison result, then $v$ is the unique viscosity solution!

$$\boxed{\text{(SP)+visc comp} \rightarrow \text{(DPP)+ } v \text{ is the unique visc sol of (DPE)}}$$

# Some comments

- the Stochastic Perron Method plus viscosity comparison substitute for (large part of) verification (in the analytic approach)
- this method represents a "probabilistic version of the analytic approach"
- loosely speaking, stochastic sub and super-solutions amount to sub and super-martingales
- stochastic sub and super-solution have to be carefully defined (depending on the control problem) as to obtain viscosity solutions as sup/inf (and to retain the comparison build in)

# Linear case

Want to compute $v(s,x) = \mathbb{E}[g(X_T^{s,x})]$, for

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_s = x. \end{cases}$$

Assumption: continuous coefficients with linear growth
There exist (possibly non-unique) weak solutions of the SDE.

$$\Big((X_t^{s,x})_{s\le t\le T}, (W_t^{s,x})_{s\le t\le T}, \Omega^{s,x}, \mathscr{F}^{s,x}, \mathbb{P}^{s,x}, (\mathscr{F}_t^{s,x})_{s\le t\le T}\Big),$$

where the $W^{s,x}$ is a $d$-dimensional Brownian motion on the stochastic basis

$$(\Omega^{s,x}, \mathscr{F}^{s,x}, \mathbb{P}^{s,x}, (\mathscr{F}_t^{s,x})_{s\le t\le T})$$

and the filtration $(\mathscr{F}_t^{s,x})_{s\le t\le T}$ satisfies the usual conditions. We denote by $\mathscr{X}^{s,x}$ the non-empty set of such weak solutions.

# Which selection of weak solutions to consider?

Just take sup/inf over all solutions.

$$v_*(s,x) := \inf_{X^{s,x} \in \mathscr{X}^{s,x}} \mathbb{E}^{s,x}[g(X_T^{s,x})]$$

and

$$v^*(s,x) := \sup_{X^{s,x} \in \mathscr{X}^{s,x}} \mathbb{E}^{s,x}[g(X_T^{s,x})].$$

The (linear) PDE associated

$$\begin{cases} -v_t - L_t v = 0 \\ v(T,x) = g(x), \end{cases} \tag{1}$$

Assumption: $g$ is bounded (and measurable).

# Stochastic sub and super-solutions

## Definition

A stochastic sub-solution of (1) $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$

1. lower semicontinuous (LSC) and bounded on $[0, T] \times \mathbb{R}^d$. In addition $u(T, x) \leq g(x)$ for all $x \in \mathbb{R}^d$.
2. for each $(s, x) \in [0, T] \times \mathbb{R}^d$, and each weak solution $X^{s,x} \in \mathscr{X}^{s,x}$, the process $(u(t, X_t^{s,x}))_{s \leq t \leq T}$ is a submartingale on $(\Omega^{s,x}, \mathbb{P}^{s,x})$ with respect to the filtration $(\mathscr{F}_t^{s,x})_{s \leq t \leq T}$.

Denote by $\mathscr{U}^-$ the set of all stochastic sub-solutions.

# Semi-solutions cont'd

Symmetric definition for stochastic super-solutions $\mathscr{U}^+$.

## Definition

A stochastic super-solution $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$

1. upper semicontinuous (USC) and bounded on $[0, T] \times \mathbb{R}^d$. In addition $u(T, x) \geq g(x)$ for all $x \in \mathbb{R}^d$.
2. for each $(s, x) \in [0, T] \times \mathbb{R}^d$, and each weak solution $X^{s,x} \in \mathscr{X}^{s,x}$, the process $(u(t, X_t^{s,x}))_{s \leq t \leq T}$ is a supermartingale on $(\Omega^{s,x}, \mathbb{P}^{s,x})$ with respect to the filtration $(\mathscr{F}_t^{s,x})_{s \leq t \leq T}$.

# About the semi-solutions

- if one choses a Markov selection of weak solutions of the SDE (and the canonical filtration), super an sub solutions are the time-space super/sub-harmonic functions with respect to the Markov process $X$
- we use the name associated to Stroock–Varadhan. In Markov framework, sub+ super-solution is a stochastic solution in the definition of Stroock-Varadhan.

The definition of semi-solutions are strong enough to provide comparison to the expectation(s).
For each $u \in \mathscr{U}^-$ and each $w \in \mathscr{U}^+$ we have

$$u \leq v_* \leq v^* \leq w.$$

Define

$$v^- := \sup_{u \in \mathscr{U}^-} u \leq v_* \leq v^* \leq v^+ := \inf_{w \in \mathscr{U}^+} w.$$

We have (need to be careful about point-wise inf)

$$v^- \in \mathscr{U}^-, \quad v^+ \in \mathscr{U}^+.$$

# Linear Stochastic Perron

### Theorem

(Stochastic Perron's Method) *If $g$ is bounded and LSC then $v^-$ is a bounded and LSC viscosity supersolution of*

$$\begin{cases} -v_t - L_t v \geq 0, \\ v(T,x) \geq g(x). \end{cases} \tag{2}$$

*If $g$ is bounded and USC then $v^+$ is a bounded and USC viscosity subsolution of*

$$\begin{cases} -v_t - L_t v \leq 0, \\ v(T,x) \leq g(x). \end{cases} \tag{3}$$

**Comment:** new method to construct viscosity solutions (recall $v^-$ and $v^+$ are anyway stochastic sub and super-solutions).

# Verification by viscosity comparison

## Definition

Condition $CP(T,g)$ is satisfied if, whenever we have a bounded (USC) viscosity sub-solution $u$ and a bounded $LSC$ viscosity super-solution $w$ we have $u \leq w$.

## Theorem

*Let $g$ be bounded and continous. Assume $CP(T,g)$. Then there exists a unique bounded and continuous viscosity solution $v$ to (1), and*

$$v_* = v = v^*.$$

*In addition, for each $(s,x) \in [0,T] \times \mathbb{R}^d$, and each weak solution $X^{s,x} \in \mathscr{X}^{s,x}$, the process $(v(t, X^{s,x}))_{s \leq t \leq T}$ is a martingale on $(\Omega^{s,x}, \mathbb{P}^{s,x})$ with respect to the filtration $(\mathscr{F}_t^{s,x})_{s \leq t \leq T}$.*

Comments:

- $v$ is a stochastic solution (in the Markov case)
- if comparison holds for all $T$ and $g$, then the diffusion is actually Markov (but we never use that explicitly)

# Idea of proof

Similar to Ishii.
To show that $v^-$ is a super-solution

- touch $v^-$ from below with a smooth test function $\varphi$
- if the viscosity super-solution property is violated, then $\varphi$ is locally a smooth sub-solution
- push it to $\varphi_\varepsilon = \varphi + \varepsilon$ slightly above, to still keep it still a smooth sub-solution (locally)
- Itô implies that $\varphi_\varepsilon$ is also (locally wrt stopping times) a submartingale along $X$
- take $\max\{v^-, \varphi_\varepsilon\}$, still a stochastic-subsolution (need to "patch" sub-martingales along a sequence of stopping times)

Comments: why don't we need Markov property? Because we only use Itô, which does not require the diffusion to be Markov.

# Obstacle problems and Dynkin games

First example of non-linear problem.

Same diffusion framework as for the linear case. Choose a selection of weak solutions $X^{s,x}$ to save on notation.

$g : \mathbb{R}^d \to \mathbb{R}$, $l, u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *bounded* and measurable, $l \le u$, $l(T,\cdot) \le g \le u(T,\cdot)$.

Denote by $\mathscr{T}^{s,x}$ the set of stopping times $\tau$ (with respect to the filtration $(\mathscr{F}_t^{s,x})_{s\le t\le T}$) which satisfy $s \le \tau \le T$.

The first player ($\rho$) *pays* to the second player ($\tau$) the amount

$$J(s,x,\tau,\rho) :=$$

$$= \mathbb{E}^{s,x}\left[\mathbb{I}_{\{\tau<\rho\}} l(\tau, X_\tau^{s,x}) + \mathbb{I}_{\{\rho\le\tau,\rho<T\}}) u(\rho, X_\rho^{s,x}) + \mathbb{I}_{\{\tau=\rho=T\}} g(X_T^{s,x})\right].$$

# Dynkin games, cont'd

*Lower value of the Dynkin game*

$$v_*(s,x) := \sup_{\tau \in \mathscr{T}^{s,x}} \inf_{\rho \in \mathscr{T}^{s,x}} J(s,x,\tau,\rho)$$

and the *upper value of the game*

$$v^*(s,x) := \inf_{\rho \in \mathscr{T}^{s,x}} \sup_{\tau \in \mathscr{T}^{s,x}} J(s,x,\tau,\rho).$$

$$v_* \leq v^*$$

Remark: we could appeal directly to what is known about Dynkin games to conclude $v_* \leq v^*$, but this is exactly what we wish to avoid.

# DPE equation for Dynkin games

$$\begin{cases} F(t, x, v, v_t, v_x, v_{xx}) = 0, \quad \text{on } [0, T) \times \mathbb{R}^d, \\ u(T, \cdot) = g, \end{cases} \tag{4}$$

where

$$\begin{aligned} F(t, x, v, v_t, v_x, v_{xx}) := \\ \max\{v - u, \min\{-v_t - L_t v, v - l\}\} \\ = \min\{v - l, \max\{-v_t - L_t v, v - u\}\}. \end{aligned} \tag{5}$$

# Super and Subsolutions

## Definition

$\mathscr{U}^+$, is the set of functions $w : [0, T] \times \mathbb{R}^d \to \mathbb{R}$

1. are continuous (C) and bounded on $[0, T] \times \mathbb{R}^d$. $w \geq l$ and $w(T, \cdot) \geq g$.
2. for each $(s, x) \in [0, T] \times \mathbb{R}^d$, and any stopping time $\tau_1 \in \mathscr{T}^{s,x}$, the function $w$ along the solution of the SDE is a super-martingale in between $\tau_1$ and the first (after $\tau_1$) hitting time of the upper stopping region $\mathscr{S}^+(w) := \{w \geq u\}$. More precisely, for any $\tau_1 \leq \tau_2 \in \mathscr{T}^{s,x}$, we have

$$w(\tau_1, X^{s,x}_{\tau_1}) \geq \mathbb{E}^{s,x}\left[w(\tau_2 \wedge \rho^+, X^{s,x}_{\tau_2 \wedge \rho^+}) | \mathscr{F}^{s,x}_{\tau_1}\right] \; - \mathbb{P}^{s,x} \; a.s.$$

where the stopping time $\rho^+$ is defined as

$$\rho^+(v, s, x, \tau_1) = \inf\{t \in [\tau_1, T] : X^{s,x}_t \in \mathscr{S}^+(w)\}.$$

**Question:** why the starting stopping time? No Markov property.

# Stochastic Perron for obstacle problems

Define symmetrically sub-solutions $\mathscr{U}^-$. Now define, again

$$v^- := \sup_{w \in \mathscr{U}^-} w \leq v_* \leq v^* \leq v^+ := \inf_{w \in \mathscr{U}^+} w.$$

Cannot show $v^- \in \mathscr{U}^-$ or $v^+ \in \mathscr{U}^+$, but it is not really needed. All is needed is stability with respect to max/min, not sup/inf (and this is the reason why we can assume continuity).

## Theorem

- *$v^-$ is viscosity super-solution of the (DPE)*
- *$v^+$ is viscosity sub-solution of the (DPE)*

# Verification by comparison for obstacle problems

## Theorem

- *if comparison holds, then there exists a unique and continuous viscosity solution $v$, equal to $v^- = v_* = v^* = v^+$*
- *the first hitting times are optimal for both players*

In the Markov case, Peskir showed (with different definitions for sub, super-solutions, which actually involve the value function) that

$$v^- = v^+$$

by showing that $v^- =$ "*value function*" $= v^+$. Peskir generalizes the characterization of value function in optimal stopping problems.

# What about optimal stopping $u = \infty$?

Classic work of El Karoui, Shiryaev: in the Markov case, the value function is the least excessive function. In our notation

$$v^+ := \inf_{w \in \mathscr{U}^+} w = v.$$

**Comment:** the proof requires to actually show that $v \in \mathscr{U}^+$. We avoid that, showing that

$$v^- \leq v \leq v^+,$$

and then using comparison.
We provide a short cut to conclude the value function is the continuous viscosity solution of the free-boundary problem (study of continuity in Bassan and Ceci)

# Back to the original control problem

work in progress

- can define the classes of stochastic super and sub-solutions such that
- the Stochastic Perron's method (existence part) works well (at least away from $T$)

Left to do:

- study the possible boundary layer at $T$
- go over verification by comparison (easy once the first step is done)

# Conclusions

- new method to construct viscosity solutions as sup/inf of stochastic sub/super-solutions
- compare directly with the value function
- if we have viscosity comparison, then the value fct is the unique continuous solution of the (DPE) and the (DPP) holds

# Conjecture

Any PDE that is associated to a stochastic optimization problem can be approached by Stochastic Perron's Method.

Games are a longer shot, but should work out.