Large systems of diffusions interacting through their ranks

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Outline

Setting

- Fix a natural number $N \in \mathbb{N}$, real numbers $b_1, b_2, \ldots, b_N$ and positive real number $\sigma_1, \sigma_2, \ldots, \sigma_N$.
- Consider a system of **interacting diffusions (particles)** on $\mathbb{R}$:

$$\mathrm{d}X_i(t) = \sum_{j=1}^{N} b_j \, \mathbf{1}_{\{X_i(t)=X_{(j)}(t)\}} \, \mathrm{d}t + \sum_{j=1}^{N} \sigma_j \, \mathbf{1}_{\{X_i(t)=X_{(j)}(t)\}} \, \mathrm{d}W_i(t),$$

where $W_1, W_2, \ldots, W_N$ are i.i.d. standard Brownian motions and some initial values $X_1(0), X_2(0), \ldots, X_N(0)$ are fixed.

Some historic remarks

- Model appeared '87 in this form in paper by **Bass** and **Pardoux** in the context of filtering theory. They proved existence and uniqueness in law.
- Model reappeared in stochastic porfolio theory ('02 book by **Fernholz**, '06 survey by **Fernholz** and **Karatzas**): diffusions $X_1, X_2, \ldots, X_N$ represent **logarithmic capitalizations**:

$$\log(\text{stock price} \times \text{number of stocks}). \tag{1}$$

Model assumes that dynamics depends only on **ranks**. **True in the long run**: explains following picture.

Capital distribution curves

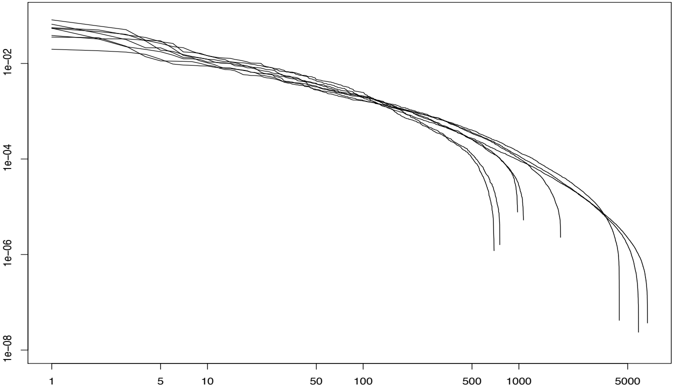

Figure: Capital distribution curves 1929-1989

Concentration results

- **Chatterjee** and **Pal** '07: for particle system above, vector of **market weights**

$$\left(\frac{e^{X_i(t)}}{\sum_{j=1}^{N} e^{X_j(t)}},\ i = 1, 2, \ldots, N\right) \tag{2}$$

  is a Markov process and its **invariant distribution concentrates** around curves of above type as $N \to \infty$. Moreover, the limit $N \to \infty$ is given by a **Poisson-Dirichlet point process** of first kind.
- **Pal**, **S.** '10 and **Ichiba**, **Pal**, **S.** '11: strong concentration of paths of market weights on any $[0, t]$ as $N \to \infty$ and fast mean-reversion as $t \to \infty$ for any fixed $N \in \mathbb{N}$.

Back to particle system

All previous results for **market weights**, which correspond to the **spacings process** in

$$\mathrm{d}X_i(t) = \sum_{j=1}^{N} b_j \, \mathbf{1}_{\{X_i(t)=X_{(j)}(t)\}} \, \mathrm{d}t + \sum_{j=1}^{N} \sigma_j \, \mathbf{1}_{\{X_i(t)=X_{(j)}(t)\}} \, \mathrm{d}W_i(t),$$

What about the **particle system** itself? Can we understand **evolution of particle density**:

$$\varrho^N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i(t)}, \quad t \geq 0.$$

Preliminaries

Here: will look at limit $N \to \infty$, which corresponds to a **hydrodynamic limit**.

First question: how to choose drift and diffusion coefficients for different N to have a meaningful limit?

Crucial observation: for **fixed** N the particle system can be written as

$$\mathrm{d}X_i(t) = b(F_{\varrho^N(t)}(X_i(t)))\,\mathrm{d}t + \sigma(F_{\varrho^N(t)}(X_i(t)))\,\mathrm{d}W_i(t)$$

for **functions** $b : [0,1] \to \mathbb{R}$, $\sigma : [0,1] \to (0,\infty)$.

$\Rightarrow$ particle system is of **mean-field** type.

Aside on mean-field models

Systems of the form

$$\mathrm{d}X_i(t) = \hat{b}(\varrho^N(t), X_i(t))\,\mathrm{d}t + \hat{\sigma}(\varrho^N(t), X_i(t))\,\mathrm{d}W_i(t) \tag{3}$$

appeared in **statistical physics**.

See: **McKean** '69, **Funaki** '84, **Oelschlager** '84, **Nagasawa, Tanaka** '85, '87, **Sznitman** '86, **Leonard** '86, **Dawson, Gärtner** '87, **Gärtner** '88.

In Gartner '88, limit $\lim_{N\to\infty} \varrho^N(t)$ is obtained under **two assumptions**.

Previous results I

Theorem. (Gärtner '88) Fix $T > 0$ and suppose $\hat{b}$, $\hat{\sigma}$ **continuous (!)**, $\hat{\sigma}$ **strictly positive**.

Let Q^N be the law of $\varrho^N(t)$, $t \in [0, T]$ on $C([0, T], M_1(\mathbb{R}))$. Then the sequence Q^N, $N \in \mathbb{N}$ is tight. Moreover, under any limit point Q^∞:

$$\forall f : \ (\varrho(t), f) - (\varrho(0), f) = \int_0^t (\varrho(s), L_{\varrho(s)} f) \, \mathrm{d}s \tag{4}$$

for all $t \in [0, T]$ almost surely. Hereby:

$$(\varrho(t), f) = \int_{\mathbb{R}} f \, \mathrm{d}\varrho(t)$$

$$L_{\varrho(s)} f = \hat{b}(\varrho(s), \cdot) f' + \frac{1}{2} \hat{\sigma}(\varrho(s), \cdot)^2 f''.$$

Previous results II

In particular, if

$$\forall f: \ (\varrho(t), f) - (\varrho(0), f) = \int_0^t (\varrho(s), L_{\varrho(s)} f) \,\mathrm{d}s \tag{5}$$

has a **unique** solution ϱ^∞ in $C([0,T], M_1(\mathbb{R}))$, then it must hold

$$\varrho^N \to \varrho^\infty, \ N \to \infty \quad \text{in probability.} \tag{6}$$

This is **not known** in general (some conditions in work of **Sznitman** '86)!

Diffusions interacting through their mean-field, intuition

How can one **guess** the **limiting dynamics**?

- Suppose we already knew $\varrho^N \to \varrho^\infty$ with ϱ^∞ **deterministic**
- Then for **large** N the system of diffusions should behave as

$$\mathrm{d}X_i(t) = \hat{b}(\varrho^\infty(t), X_i(t))\,\mathrm{d}t + \hat{\sigma}(\varrho^\infty(t), X_i(t))\,\mathrm{d}W_i(t) \tag{7}$$

- Thus, the empirical measure converges to the law of

$$\mathrm{d}X(t) = \hat{b}(\varrho^\infty(t), X(t))\,\mathrm{d}t + \hat{\sigma}(\varrho^\infty(t), X(t))\,\mathrm{d}W(t) \tag{8}$$

- **Ito's formula** for $f(X(t))$ and $\mathscr{L}(X(t)) = \varrho^\infty(t)$ imply:

$$(\varrho^\infty(t), f) - (\varrho^\infty(0), f) = \int_0^t (\varrho^\infty(s), L_{\varrho^\infty(s)} f)\,\mathrm{d}s$$

Previous results III

Theorem. (Dawson, Gärtner '87) Fix $T > 0$ and suppose $\hat{b}$ **continuous**, $\hat{\sigma} \equiv 1$ **(!)**

Then the sequence $(\varrho^N(t),\ t \in [0, T]),\ N \in \mathbb{N}$ satisfies a large deviations principle on $C([0, T], M_1(\mathbb{R}))$ with the good rate function

$$I(\gamma) = \sup_{g \in \overline{\mathcal{S}}} \left[(\gamma(T), g) - (\gamma(0), g) - \int_0^T (\gamma(t), \mathcal{R}_t^\gamma g + \frac{1}{2}(g_x)^2)\,\mathrm{d}t \right]$$

and scale N. Hereby:

$$\mathcal{R}_t^\gamma g = g_t + \hat{b}(\varrho(s), \cdot) g_x + \frac{1}{2} g_{xx}.$$

Relation to Burger's equation

Remarks:

- **Goodness** of rate function + **LDP** imply: ϱ^N will concentrate around the set $\{\gamma : I(\gamma) = 0\}$.
- If we apply this result with $\hat{b}(\varrho(s), \cdot) = -F_{\varrho(s)}(\cdot)$ (**discontinuity!**), then only point with $I(\gamma) = 0$ is the one, whose path of cdfs $R(t, \cdot) = F_{\gamma(t)}(\cdot)$ is the unique weak solution of **viscous Burger's equation:** $R_t = -\frac{1}{2}(R^2)_x + \frac{1}{2} R_{xx}$.

  I.e.: **particle system approximation** of R. Same result for a particle system with local time interactions in **Sznitman** '86.

Our results I

Theorem 1. (Dembo, Krylov, S., Varadhan, Zeitouni '12) Fix $T > 0$ and suppose $\hat{b}(\varrho(t), x) = b(F_{\varrho(t)}(x))$, $\hat{\sigma} = \sigma(F_{\varrho(t)}(x))$; b and $A := \frac{1}{2}\sigma^2$ nice. Then, $(\varrho^N(t),\ t \in [0, T])$, $N \in \mathbb{N}$ satisfies an LDP on $C([0, T], M_1(\mathbb{R}))$ with scale N and good rate function J defined by

$$J(R) = \frac{1}{2}\left\| \frac{\sigma(R)}{2}\, \frac{R_t - (A(R)R_x)_x}{A(R)(R_x)^{1/2}} - \frac{b(R)}{\sigma(R)}(R_x)^{1/2} \right\|^2_{L^2(\mathbb{R}_T)}$$

for all $R \in C_b(\mathbb{R}_T)$ with $R_t, R_x, R_{xx} \in L^{3/2}(\mathbb{R}_T)$, $R_x \in L^3(\mathbb{R}_T)$, R_x having finite $(1+\varepsilon)$ moment, $t \mapsto (R_x(t,\cdot), g(t,\cdot))$ abs. cont.; $J = \infty$ otherwise. Hereby, $R = F_{\gamma(\cdot)}(\cdot)$.

Our results II

Consequences:

- **Goodness of rate function** and **LDP** imply that ϱ^N concentrates around the set $\{\gamma : \; J(\gamma) = 0\}$.
- The only path γ with $J(\gamma) = 0$ corresponds to the unique weak solution of the **generalized porous medium equation with convection:** $R_t = \Sigma(R)_{xx} + \Theta(R)_x$.
- Hence, we found a **particle system approximation** for the solution of the latter, which converges exponentially fast.

Our results III

In the course of the proof show the following **regularity result** in nonlinear PDEs:

Theorem 2. Consider a weak solution of the Cauchy problem for the **tilted generalized porous medium equation**:

$$R_t - (A(R)R_x)_x = h\,A(R)\,R_x, \quad R(0,\cdot) = R_0.$$

such that $R \in C_b(\mathbb{R}_T)$ and $R_x(t,\cdot)\,\mathrm{d}x$ is a probability measure for every t. If $\int_{\mathbb{R}_T} h^2\,R_x\,\mathrm{d}m < \infty$ and R_x has finite $(1+\varepsilon)$ moment, then R_t, R_x, R_{xx} exist as elements of $L^{3/2}(\mathbb{R}_T)$, $R_x \in L^3(\mathbb{R}_T)$ and

$$\int_{\mathbb{R}_T} \frac{R_{xx}^2}{R_x}\,\mathrm{d}m < \infty, \quad \int_{\mathbb{R}_T} \frac{R_t^2}{R_x}\,\mathrm{d}m < \infty.$$

Proof sketch, general principles I

Localization: LDP holds, if we can show **weak/local LDP**:

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(\varrho^N \in C) \le - \inf_{\gamma \in C} J(\gamma) \ \text{ for all compacts } C,$$
$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}(\varrho^N \in U) \ge - \inf_{\gamma \in U} J(\gamma) \ \text{ for all open sets } U$$

and **exponential tightness**:

$$\forall K > 0 \, \exists C_K \text{ compact} : \ \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(\varrho^N \notin C_K) \le -K.$$

Proof sketch, general principles II

Alternative characterization of **weak/local LDP**:

$$\forall\gamma:\ \lim_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}(\varrho^N\in B(\gamma,\delta))\leq -J(\gamma),$$
$$\forall\gamma:\ \lim_{\delta\downarrow 0}\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}(\varrho^N\in B(\gamma,\delta))\geq -J(\gamma)$$

What we prove:

- Local upper bound holds with a **Dawson-Gärtner type rate function I**
- Local lower bound holds with the **desired rate function J**
- $J\leq I$.

Proof sketch, upper bound around a path γ

Appropriate **variational problem**:

$$I(\gamma) = \sup_{g\in\overline{\mathcal{S}}} \Big[(\gamma(T), g) - (\gamma(0), g) - \int_0^T (\gamma(t), \mathcal{R}_t^\gamma g + \frac{1}{2}A(R)(g_x)^2)\,\mathrm{d}t\Big],$$

where $\mathcal{R}_t^\gamma g = g_t + b(R)g_x + A(R)g_{xx}$, $R = F_{\gamma(\cdot)}(\cdot)$.

Why **appropriate**? On the event $\varrho^N \in B(\gamma,\delta)$, our particle system is **close** to solution of

$$\mathrm{d}Y_i(t) = b(F_{\gamma(t)}(Y_i(t)))\,\mathrm{d}t + \sigma(F_{\gamma(t)}(Y_i(t)))\,\mathrm{d}W_i(t),\ i = 1, 2, \ldots, N.$$

on **exponential scale**.

Proof sketch, upper bound around a path γ

Pick test function g, apply **Itô's formula**:

$$\mathrm{d}g(t, Y_i(t)) = (g_t + b(R)g_x + A(R)g_{xx})(t, Y_i(t))\,\mathrm{d}t \\ + g_x(t, Y_i(t))\sigma(F_{\gamma(t)}(Y_i(t)))\,\mathrm{d}W_i(t).$$

Hence,

$$\mathrm{d}(\varrho_Y^N(t), g(t,\cdot)) = (\varrho_Y^N(t), g_t + b(R)g_x + A(R)g_{xx})\,\mathrm{d}t \\ + \frac{1}{N}\sum_{i=1}^{N} g_x(t, Y_i(t))\sigma(F_{\gamma(t)}(Y_i(t)))\,\mathrm{d}W_i(t).$$

Note: **martingale part of order $\frac{1}{N}$**. **Freidlin-Wentzell** type problem!

Proof sketch, upper bound around a path γ

Thus, for **fixed** g, rate is given by

$$I^g(f) = \frac{1}{2}\int_0^T \frac{\left|\dot{f}(u) - \left(\gamma(u), g_t + b(R)g_x + A(R)g_{xx}\right)\right|^2}{(\gamma(u), \sigma(R)^2(g_x)^2)}\,\mathrm{d}u.$$

We are interested in $f(t) = (\gamma(t), g(t,\cdot))$. Plug it in, integrate by parts, take sup: **upper bound** with

$$I(\gamma) = \sup_{g\in\overline{\mathcal{S}}} I^g((\gamma(t), g(t,\cdot))).$$

Done with **local UBD**!

Proof sketch, lower bound around a path γ

Consider a γ such that $J(\gamma) < \infty$. Then, view $R = F_{\gamma(\cdot)}(\cdot)$ as solution of

$$R_t - (A(R)R_x)_x = h\,A(R)\,R_x.$$

That is, **set** $h = \frac{R_t - (A(R)R_x)_x}{A(R)R_x}$. This form allows for a **tilting argument**:

Main idea: apply **Girsanov's Theorem** to change particle system to:

$$\mathrm{d}X_i(t) = -h(t, X_i(t))A(F_{\varrho^N(t)}(X_i(t)))\,\mathrm{d}t + \sigma(F_{\varrho^N(t)}(X_i(t)))\,\mathrm{d}\tilde{W}_i(t),$$

then show $\frac{\mathrm{d}P^N}{\mathrm{d}\tilde{P}^N} \approx e^{-N\,J(\gamma)}$ on $\{\varrho^N \in B(\gamma, \delta)\}$ and **LLN** under $\tilde{P}^N$: $\lim_{N\to\infty} \varrho^N = \gamma$.

Proof sketch, lower bound around a path γ

The proof of **LLN** in the usual way:

- First, show **tightness** of ϱ^N, $N \in \mathbb{N}$.
- Then, show every limit point $\tilde{\gamma}$ corresponds to a weak solution of

$$\tilde{R}_t - (A(\tilde{R})\tilde{R}_x)_x = h\, A(\tilde{R})\, \tilde{R}_x$$

  via $\tilde{R} = F_{\tilde{\gamma}(\cdot)}(\cdot)$.

- Finally, show that **weak solution** of PDE **unique**, thus: $\gamma = \tilde{\gamma}$.

Technical point: for Girsanov, tightness, passing to the limit, uniqueness need: $h \in C_b(\mathbb{R}_T)$, Lipschitz.

Proof sketch, lower bound around a path γ

What do we mean by $\frac{\mathrm{d}P^N}{\mathrm{d}\tilde{P}^N} \approx e^{-N J(\gamma)}$ on $\{\varrho^N \in B(\gamma, \delta)\}$?

- By **Girsanov's Theorem**:

$$P(\varrho^N \in B(\gamma, \delta)) = \mathbb{E}^{\tilde{P}}\left[e^{M(T) - \langle M \rangle (T)/2} \mathbf{1}_{\{\varrho^N \in B(\gamma,\delta)\}}\right]$$

- Apply **Hölder's inequality** to lower bound $P(\varrho^N \in B(\gamma, \delta))$ by:

$$\mathbb{E}^{\tilde{P}}\left[e^{-\frac{q}{p}M(T) + \frac{q}{p}\langle M \rangle (T)/2}\right]^{-p/q} \tilde{P}(\varrho^N \in B(\gamma, \delta))^p.$$

Proof sketch, lower bound around a path γ

- Next, **complete the martingale**:

$$\mathbb{E}^{\tilde{P}}\left[e^{-\frac{q}{p}M(T)+\frac{q}{2p}\langle M\rangle(T)}\right] = \mathbb{E}^{\tilde{P}}\left[e^{-\frac{q}{p}M(T)-\frac{q^2}{2p^2}\langle M\rangle(T)+(\frac{q}{2p}+\frac{q^2}{2p^2})\langle M\rangle(T)}\right].$$

- Finally,

$$N(J(\gamma)-\varepsilon) \leq \langle M\rangle(T) \leq N(J(\gamma)+\varepsilon),$$

since we work under $\tilde{P}$ now! Remains to take limits $N \to \infty$, $\delta \downarrow 0$, $p \uparrow \infty$, $q \downarrow 1$, $\varepsilon \downarrow 0$. Done with **local LBD**!

Proof sketch, comparison of rate functions I

What did we prove? local UBD with I, **local LBD** with J.

Need to show: $J \leq I$:

- Fix γ. Can assume $I(\gamma) < \infty$.
- Use $I(\gamma) < \infty$ to deduce regularity of $R = F_{\gamma(\cdot)}(\cdot)$:

  $R_t, R_x, R_{xx} \in L^{3/2}(\mathbb{R}_T)$, $R_x \in L^3(\mathbb{R}_T)$, $\int_{\mathbb{R}_T} \frac{R_{xx}^2}{R_x}\,\mathrm{d}m < \infty$, $\int_{\mathbb{R}_T} \frac{R_t^2}{R_x}\,\mathrm{d}m < \infty$.
- Recall I defined as supremum over $g \in \overline{\mathcal{S}}$ of

$$\Big[(\gamma(T), g) - (\gamma(0), g) - \int_0^T (\gamma(t), g_t + A(R)g_{xx} + \frac{1}{2}A(R)(g_x)^2)\,\mathrm{d}t\Big],$$

  would like to take $g_x = \frac{R_t - (A(R)R_x)_x}{A(R)R_x} = h$.

Proof sketch, comparison of rate functions II

- This is OK due to **regularity:** $h \in L^2(\mathbb{R}_T, R_x)$ and denseness of $\overline{\mathcal{S}}$ in the latter.
- Done, since $J(\gamma) = \frac{1}{4}\int_{\mathbb{R}_T} h^2\, R_x \,\mathrm{d}m$.
- Now, redo this **with drift** and end up with

$$J(R) = \frac{1}{2}\left\| \frac{\sigma(R)}{2} \frac{R_t - (A(R)R_x)_x}{A(R)(R_x)^{1/2}} - \frac{b(R)}{\sigma(R)}(R_x)^{1/2} \right\|^2_{L^2(\mathbb{R}_T)}$$

as desired.

Under the rug

- **Regularity results** from $I(\gamma) < \infty$.
- **Getting rid of the atoms**: used continuity of R at various places, e.g. uniqueness of solutions to tilted generalized porous medium equation.
- **Uniqueness of weak solutions to tilted PME**.
- **Exponential tightness**.

THANK YOU
FOR YOUR ATTENTION!