Some sufficient condition for the ergodicity of the Lévy transform

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Lévy transformation of the path space

- β is a Brownian motion

$$\mathbf{T}\beta = \int \operatorname{sign}(\beta_s) d\beta_s = |\beta| - L^0(\beta).$$

- $\mathbf{T}$ is a transformation of the path space.
- $\mathbf{T}$ preserves the Wiener measure.
- Is $\mathbf{T}$ ergodic?
- A deep result of Marc Malric claims that the Lévy transform is topologically recurrent, i.e., on an almost sure event

$$\{\mathbf{T}^n\beta \,:\, n \geq 0\} \cap G \neq \emptyset, \quad \text{for all nonempty open } G \subset C[0,\infty).$$

- We use only a weaker form, also due to Marc Malric, the density of zeros of iterated paths, i.e.:

$$\bigcup_{n=0}^{\infty} \{\, t > 0 \,:\, (\mathbf{T}^n\beta)_t = 0 \} \text{ is dense in } [0,\infty).$$

Ergodicity and Strong mixing (reminder)

$$T : \Omega \to \Omega, \quad \mathbf{P} \circ T^{-1} = \mathbf{P}$$

- T is ergodic, if
  - $\frac{1}{N} \sum_{n=0}^{N-1} \mathbf{P}\left(A \cap T^{-n} B\right) \to \mathbf{P}(A)\, \mathbf{P}(B), \quad$ for all A, B.
  - or, $\frac{1}{N} \sum_{n=0}^{N-1} X \circ T^{n} \to \mathbf{E}(X), \quad$ for each r.v. $X \in L^1$.
  - or, the invariant σ–field, is trivial.
- T is strongly mixing if $\mathbf{P}\left(A \cap T^{-n} B\right) \to \mathbf{P}(A)\, \mathbf{P}(B), \quad$ for all A, B.

Ergodicity and weak convergence

In our case $\Omega = C[0, \infty)$ is a polish space (complete, separable, metric space).

Theorem

Ω polish, T is a measure preserving transform of $(\Omega, \mathcal{B}(\Omega), \mathbf{P})$. Then

- *T is ergodic iff* $\frac{1}{n} \sum_{k=0}^{n-1} \mathbf{P} \circ (T^0, T^k)^{-1} \xrightarrow{w} \mathbf{P} \otimes \mathbf{P}$ *as $n \to \infty$.*
- *T is strongly mixing iff* $\mathbf{P} \circ (T^0, T^n)^{-1} \xrightarrow{w} \mathbf{P} \otimes \mathbf{P}$ *as $n \to \infty$.*

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- *T is strongly mixing iff* $\mathbf{P} \circ (T^0, T^n)^{-1} \xrightarrow{w} \mathbf{P} \otimes \mathbf{P}$ *as* $n \to \infty$.

Note that both families of measures are tight:

$$\left\{\mathbf{P} \circ (T^0, T^n)^{-1} \,:\, n \geq 0\right\} \quad \text{and} \quad \left\{\frac{1}{n}\sum_{k=0}^{n-1} \mathbf{P} \circ (T^0, T^n)^{-1} \,:\, n \geq 0\right\}$$

If $C \subset \Omega$ compact, with $\mathbf{P}(\Omega \setminus C) < \varepsilon$ then
$\mathbf{P}\left((T^0, T^k) \notin C \times C\right) \leq \mathbf{P}\left(T^0 \notin C\right) + \mathbf{P}\left(T^k \notin C\right) < 2\varepsilon.$

Convergence of finite dim. marginals ($\xrightarrow{f.d.}$) is enough

Some notations:

- β is the canonical process on $\Omega = C[0,\infty)$,
- $h : [0,\infty) \times C[0,\infty)$ progressive, $|h| = 1 \ dt \otimes d\mathbf{P}$ a.e.

$$T : \Omega \to \Omega, \qquad T\beta = \int_0^{\cdot} h(s,\beta) d\beta_s, \quad \text{(e.g. } h(s,\beta) = \operatorname{sign}(\beta_s)\text{)}.$$

- $\beta^{(n)} = T^n \beta$ is the n–th iterated path.
- $h^{(0)} = 1$, $h_s^{(n)} = \prod_{k=0}^{n-1} h(s, \beta^{(k)})$ for $n > 0$, so $\beta_t^{(n)} = \int_0^t h_s^{(n)} d\beta_s$.

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Then

- The distribution of $(\beta,\beta^{(n)})$ is $\mathbf{P} \circ (T^0, T^n)^{-1}$
- Let κ_n is uniform on $\{0,1,\dots,n-1\}$ and independent of β. The law of $(\beta,\beta^{(\kappa_n)})$ is $\frac{1}{n}\sum_{k=0}^{n-1} \mathbf{P} \circ (T^0, T^k)^{-1}$.
- T is strongly mixing, iff $(\beta,\beta^{(n)}) \xrightarrow{f.d.}$ BM–2.
- Similarly, T is ergodic, iff $(\beta,\beta^{(\kappa_n)}) \xrightarrow{f.d.}$ BM–2.
- Reason: Tightness + $\xrightarrow{f.d.}$ = convergence in law.

Characteristic function

- Fix $t_1, \ldots, t_r \geq 0$ and $\alpha = (a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{R}^{2r}$
- The characteristic function of $(\beta_{t_1}, \ldots, \beta_{t_r}, \beta^{(n)}_{t_1}, \ldots, \beta^{(n)}_{t_r})$ at α is

$$\varphi_n = \mathbf{E}\left(e^{i\left(\int f(s)d\beta_s + \int g(s)d\beta^{(n)}_s\right)} \right) = \mathbf{E}\left(e^{i\int \left(f(s)+g(s)h^{(n)}_s\right)d\beta_s} \right),$$

  where $f = \sum_{j=1}^r a_j \mathbb{1}_{[0,t_j]}$, $g = \sum_{j=1}^r b_j \mathbb{1}_{[0,t_j]}$.
- Finite dim. marginals has the right limit, if for all choices $r \geq 1$, $\alpha \in \mathbb{R}^{2r}$, $t_1, \ldots, t_r \geq 0$

$$\varphi_n \to \exp\left\{ -\frac{1}{2}\int f^2 + g^2 \right\} \quad \text{for strong mixing,}$$

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi_k \to \exp\left\{ -\frac{1}{2}\int f^2 + g^2 \right\} \quad \text{for ergodicity.}$$

Estimate for $|\varphi_n - \varphi|$, where $\varphi = e^{-\frac{1}{2}\int f^2+g^2}$

- $M_t = \int_0^t (f(s) + h_s^{(n)} g(s)) d\beta_s$.
- M is a closed martingale and so is $Z = \exp\left\{ iM + \frac{1}{2}\langle M \rangle \right\}$.
- $Z_0 = 1 \implies \mathbf{E}(Z_\infty) = 1$.
- $\langle M \rangle_\infty = \int_0^\infty f^2(s) + g^2(s) ds + 2\int_0^\infty h_s^{(n)} f(s) g(s) ds$
- $$\varphi = \varphi \mathbf{E}(Z_\infty) = \mathbf{E} \exp\left\{ i \int_0^\infty (f d\beta + g d\beta^{(n)}) + \int_0^\infty fgh^{(n)} \right\}$$
- Recall that $fg = \sum_j a_j b_j \mathbb{1}_{[0,t_j]}$. Then with $X_n(t) = \int_0^t h_s^{(n)} ds$

$$|\varphi_n - \varphi| \leq \mathbf{E}\left| 1 - e^{\int_0^\infty fgh^{(n)}} \right| \leq e^{\int |fg|} \mathbf{E}\left| \int_0^\infty fgh^{(n)} \right| \leq C \sum_{j=1}^r \mathbf{E}\left| X_n(t_j) \right|,$$

where $C = C(f, g) = C(\alpha, t_1, \ldots, t_r)$ does not depend on n.

$X_n(t) = \int_0^t h_s^{(n)} ds \xrightarrow{p} 0$ for all $t \geq 0$ would be enough

Theorem

1. *If $X_n(t) \xrightarrow{p} 0$ for all $t \geq 0$, then T is strongly mixing.*
2. *T is ergodic if and only if $\frac{1}{n} \sum_{k=0}^{n-1} X_k^2(t) \xrightarrow{p} 0$ for all $t \geq 0$.*

Strong mixing:

- The only missing part is the convergence of finite dimensional marginals.
- If $X_n(t) \xrightarrow{p} 0$ then $\mathbf{E}\,|X_n(t)| \to 0$ since $|X_n(t)| \leq t$.
- Then $|\varphi_n - \varphi| \leq C \sum_j \mathbf{E}\,|X_n(t_j)| \to 0 \implies (\beta, \beta^{(n)}) \xrightarrow{f.d.}$ BM-2.

Remember, that:

- $(\beta, \beta^{(n)}) \xrightarrow{f.d.}$ BM-2 + tightness gives: $(\beta, \beta^{(n)}) \xrightarrow{D}$ BM-2.
- $(\beta, \beta^{(n)}) \xrightarrow{D}$ BM-2 $\Leftrightarrow$ T strong mixing.

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Ergodicity. $\Leftarrow$.

- By Cauchy-Schwarz and $|X_k(t)| \leq t$

$$\mathbf{E}\left(\frac{1}{n} \sum_{k=0}^{n-1} |X_k(t)| \right) \leq \mathbf{E}^{1/2} \left(\frac{1}{n} \sum_{k=0}^{n-1} X_k^2(t) \right) \to 0.$$

- Then $\left| \frac{1}{n} \sum_{k=0}^{n-1} \varphi_k - \varphi \right| \leq C \sum_j \mathbf{E} \frac{1}{n} \sum_{k=0}^{n-1} |X_k(t_j)| \to 0 \implies$ $(\beta, \beta^{(\kappa_n)}) \xrightarrow{f.d.}$ BM-2.

Remeber that

- κ_n is uniform on $\{0, \ldots, n-1\}$ and independent of β
- $(\beta, \beta^{(\kappa_n)}) \xrightarrow{f.d.}$ BM-2 + tightness gives: $(\beta, \beta^{(\kappa_n)}) \xrightarrow{D}$ BM-2.
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Ergodicity. $\Rightarrow$ (outline of the proof)

- Fix $0 < s < t$. Then the following limits exist a.s and in L^2:

$$Z_u = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h_s^{(k)} h_u^{(k)} \quad \text{for } s \leq u \leq t, \qquad Z = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h_s^{(k)} (\beta_t^{(k)} - \beta_s^{(k)})$$

  moreover $|Z|$ and $|Z_u|$ are invariant for T, hence they are non-random.

- Then $h_s^{(k)}(\beta_t^{(k)} - \beta_s^{(k)}) = \int_s^t h_s^{(k)} h_u^{([k)} d\beta_u$ and

$$Z = \int_s^t Z_u d\beta_u = \int_s^t |Z_u| \, d\tilde{\beta}_u, \quad \text{where} \quad \tilde{\beta} = \int_s^{\cdot} \text{sign}(Z_u) d\beta_u.$$

- $Z \sim N(0, \sigma^2)$ since $|Z_u|$ is non-random. But $|Z|$ is also non-random.
  $\Longrightarrow \; Z = 0. \; \Longrightarrow \; Z_u = 0. \; \Longrightarrow \; \frac{1}{n}\sum_{k=0}^{n-1} X_k^2(t) \to 0.$

A variant of the mean ergodic theorem

- T is a measure preserving transformation of Ω,
- ε_0 is r.v. taking values in $\{-1,+1\}$, $\varepsilon_k = \varepsilon_0 \circ T^k$.
- For $\xi \in L^2(\Omega)$, $U\xi = \xi \circ T\varepsilon_0$ is an isometry.
- von Neumann's mean errgodic theorem says, that

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k\xi \to P\xi, \in L^2$$

where P is the projection onto

$$\left\{X \in L^2 \,:\, X \circ T\varepsilon_0 = UX = X\right\}$$

- $|P\xi|$ is invariant under T.
- what is $U^k\xi$?

$$U\xi = \xi \circ T\varepsilon_0, \quad U^2\xi = \xi \circ T^2\varepsilon_1\varepsilon_0, \quad \dots \quad U^k\xi = \xi \circ T^k \prod_{j=0}^{k-1}\varepsilon_j,$$

- Almost sure convergence also holds by the subadditive ergodic theorem.

Lévy transformation

- The Lévy transformation $\mathbf{T}$ is scaling invariant, that is, if for $x > 0$ $\Theta_x : C[0,\infty) \to C[0,\infty)$ denotes $\Theta_x(w)(t) = xw(t/x^2)$ then

$$\Theta_x \mathbf{T} = \mathbf{T}\Theta_x$$

- As before $\beta^{(n)} = \beta \circ \mathbf{T}^n$, $h_t^{(n)} = \prod_{k=0}^{n-1} \operatorname{sign}(\beta_t^{(k)})$, $X_n(t) = \int_0^t h_s^{(n)} ds$.

By scaling we get:

Theorem

1. *If $X_n(1) \xrightarrow{p} 0$ as $n \to \infty$, then $\mathbf{T}$ is strongly mixing.*
2. *$\mathbf{T}$ is ergodic, if and only if $\frac{1}{n} \sum_{k=0}^{n-1} X_k^2(1) \xrightarrow{p} 0$ as $n \to \infty$.*

Behaviour of $X_n(1)$

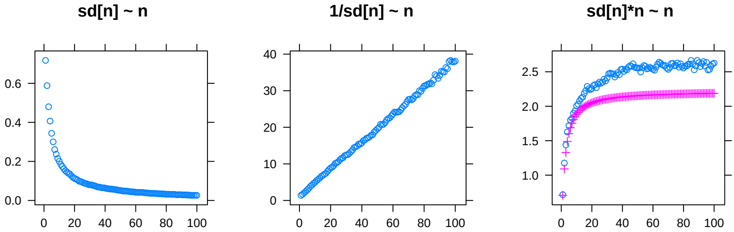

Figure: $sd^2[n] = \mathbf{E}\left(X_n^2(1)\right)$, sample size: 2000, number of steps of SRW: 10^6. Pink crosses denotes $n \times \mathbf{E}^{1/2}\left(\tilde{X}_n^2\right)$, where $\tilde{X}_n = \int_0^1 \prod_{k=0}^{n-1} \text{sign}(\tilde{\beta}_s^{(k)}) ds$ with independent BM-s $(\tilde{\beta}^{(k)})_{k\geq 0}$. We have $\mathbf{E}^{1/2}\left(\tilde{X}_n^2\right) \approx \frac{\pi}{n\sqrt{2}} \approx \frac{2.22}{n}$.

Conjecture

$\mathbf{E}\left(X_n^2(1)\right) = O(1/n^2)$. *This would give:* $X_n(1) = \int_0^1 h_s^{(n)} ds \to 0$ *almost surely.*

Simplification

- Goal: $X_n(1) = \int_0^1 h_s^{(n)} ds \xrightarrow{p} 0$.
- Enough: $X_n(1) \to 0$ in L^2.
- $$\mathbf{E}\left(X_n^2(1)\right) = 2\int_{0<u<v<1} \mathbf{E}\left(h_u^{(n)} h_v^{(n)}\right) dudv = 2\int_{0<u<v<1} \mathrm{cov}\left(h_u^{(n)}, h_v^{(n)}\right) dudv.$$
- Enough:
$$\mathbf{E}\left(h_s^{(n)} h_1^{(n)}\right) \to 0, \quad \text{for } 0 < s < 1,$$
by boundedness and scaling.
- New goal: fixing $s \in (0,1)$,
$$\mathbf{P}\left(h_s^{(n)} h_1^{(n)} = 1\right) \approx \mathbf{P}\left(h_s^{(n)} h_1^{(n)} = -1\right), \quad \text{for } n \text{ large.}$$

Idea: coupling.

Coupling I.

- Assume that $S : C[0,\infty) \to C[0,\infty)$ preserves $\mathbf{P}$.
- Denote by $\tilde{\beta}^{(n)}$ the shadow path $\beta^{(n)} \circ S$.
- Assume also that there is an event A such that on A the sequences

$$\operatorname{sign}(\beta_s^{(n)})\operatorname{sign}(\beta_1^{(n)}) \quad \text{and} \quad \operatorname{sign}(\tilde{\beta}_s^{(n)})\operatorname{sign}(\tilde{\beta}_1^{(n)})$$

  differ at exactly one index denoted by ν.

Then

$$\limsup_{n\to\infty} \left| \mathbf{E}\left(h_s^{(n)} h_1^{(n)} \right) \right| \leq \mathbf{P}\left(A^c\right).$$

Reason:

$$\left| \mathbf{E}\, h_s^{(n)} h_1^{(n)} \right| = \left| \mathbf{E}\left(\frac{h_s^{(n)} h_1^{(n)} + \tilde{h}_s^{(n)} \tilde{h}_1^{(n)}}{2} \right) \right| \leq \mathbf{P}(A^c) + \mathbf{P}\left(n < \nu\right).$$

Coupling.

Proposition

If there is a stopping time τ, s.t.

- $s < \tau < 1$,
- *exists* $\nu < \infty$, *s.t.* $\beta_\tau^{(\nu)} = 0$,
- $\min_{0 \le k < \nu} |\beta_\tau^{(k)}| > C\sqrt{1-\tau}$,

$$\Longrightarrow \quad \limsup_n \left| \mathbf{E}\, h_s^{(n)} h_1^{(n)} \right| \le \mathbf{P}\left(\max_{s \in [0,1]} |\beta_s| > C \right).$$

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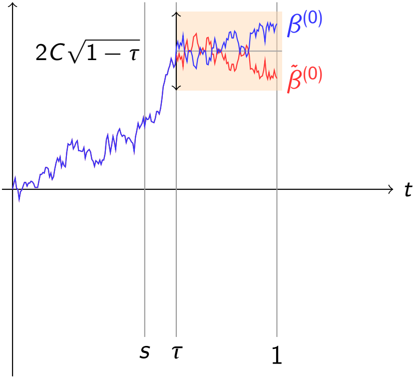
- S reflects β after τ:

$$(S\beta)_t = \tilde{\beta}_t = \beta_{t\wedge\tau} - (\beta_t - \beta_{t\wedge\tau}).$$

- $A = \left\{ \max_{t\in[\tau,1]} |\beta_t^{(0)} - \beta_\tau^{(0)}| \le C\sqrt{1-\tau} \right\}$.
- Then

$$\mathbf{P}(A^c) = \mathbf{P}\left(\max_{s\in[0,1]} |\beta_s| > C \right)$$

by strong Markov property and scaling.



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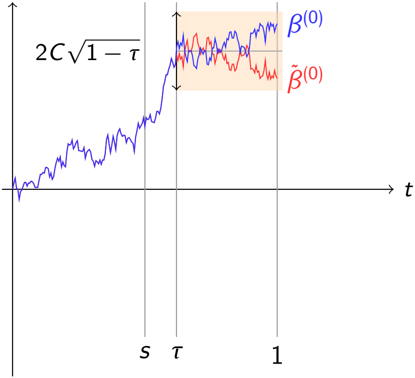
- We need that

  $\operatorname{sign}(\beta_s^{(n)}\beta_1^{(n)})$ differs from $\operatorname{sign}(\tilde{\beta}_s^{(n)}\tilde{\beta}_1^{(n)})$

  at exactly one place, when $n = \nu$.
- Recall that $\mathbf{T}\beta = |\beta| - L$.



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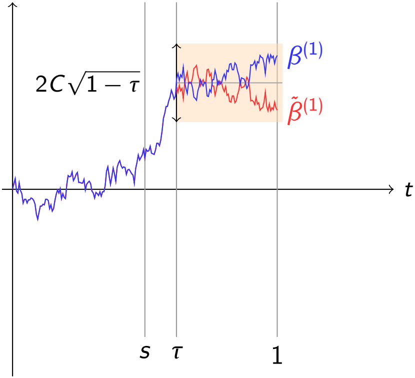

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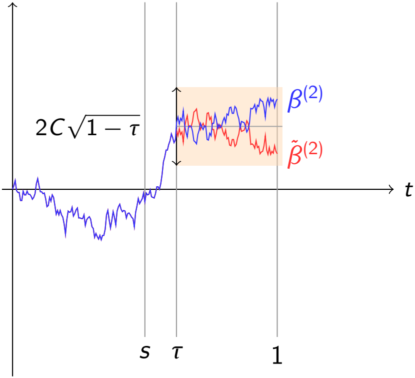

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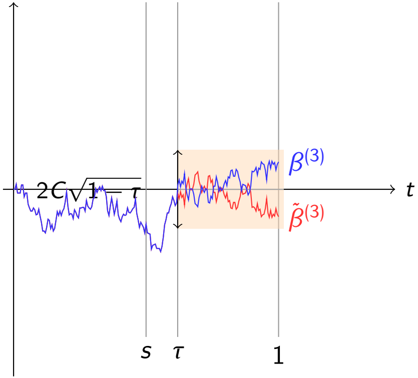

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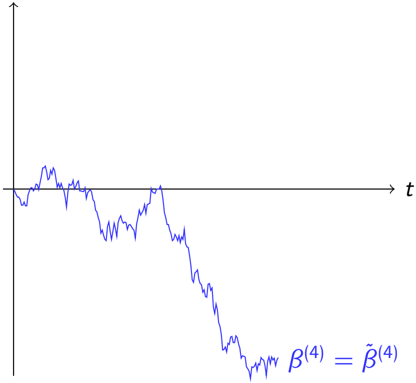
  $$\mathrm{sign}(\beta_s^{(n)}\beta_1^{(n)}) \quad \text{differs from} \quad \mathrm{sign}(\tilde\beta_s^{(n)}\tilde\beta_1^{(n)})$$

  at exactly one place, when $n = \nu$.
- Recall that $\mathbf{T}\beta = |\beta| - L$.



Simplification

Proposition

If there is a random time τ, s.t,

1. $s < \tau < 1$,
2. *exists* $\nu < \infty$, *s.t.* $\beta_\tau^{(\nu)} = 0$,
3. $\min_{0 \leq k < \nu} |\beta_\tau^{(k)}| > C\sqrt{1-\tau}$,

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$$\tau_n = \inf\left\{ t \ge s \,:\, \beta_t^{(n)} = 0,\ \min_{0\le k<n} |\beta_t^{(k)}| \ge C\sqrt{(1-t)\vee 0} \right\}, \quad \tilde\tau = \inf_n \tau_n.$$

- τ_n, $\tilde\tau$ are stopping times.
- By the condition $s \le \tilde\tau < 1$.
- If for some $\omega\in\Omega$, $\tilde\tau(\omega) < \tau_n(\omega)$ for all n then by continuity

$$\inf_{n\ge 0} |\beta_{\tilde\tau}^{(n)}| \ge C\sqrt{1-\tilde\tau} > 0 \quad \text{at } \omega.$$

- This can only happen with probability zero due to Malric's density theorem!!

Good time points

Definition

For $s \in (0,1)$, $C > 0$

$$A(C,s) = \left\{ t \geq 0 \,:\, \exists \gamma, n,\, s \cdot t < \gamma < t,\, \beta_\gamma^{(n)} = 0, \right.$$
$$\left. \min_{0 \leq k < n} \left| \beta_\gamma^{(n)} \right| > C\sqrt{t - \gamma} \right\}$$

is the set of good time points.
That is, t is good, if some iterated path has a zero close to t and previous iterates are sufficiently large.

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Goal:

$$\mathbf{P}\left(1 \in A(C,s)\right) = 1, \quad \text{for all } C > 0,\ s \in (0,1).$$

Set of good points II.

$$A(C,s) = \Big\{ t \geq 0 \, : \, \exists \gamma, n, \; s \cdot t < \gamma < t, \; \beta_\gamma^{(n)} = 0,$$

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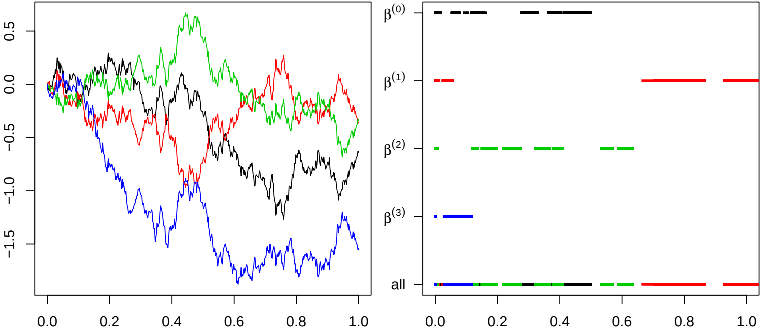
- $\mathbf{P}\left(t \in A(C,s)\right)$ does not depend on t.
- $\mathbf{P}\left(1 \in A(C,s)\right) = 1 \Leftrightarrow A(C,s)$ has full Lebesgue measure almost surely
  Proof: Let Z exponential independent of $\beta^{(0)}$. Then

$$1 = \mathbf{P}\left(Z \in A(C,s)\right) = \int_0^\infty \mathbf{P}\left(t \in A(C,s)\right) e^{-t} dt.$$

New goal:
The random set of good time points $A(C,s)$ is of full Lebesgue measure almost surely.

Good time points, a picture $s = .9$ and $C = 2$



If γ is a zero of $\beta^{(n)}$ and $\min_{0 \le k < n} |\beta^{(k)}_\gamma| = \xi > 0$ then

$$I = (\gamma, \gamma + L) \subset A(C, s), \quad \text{where} \quad L = \frac{\xi^2}{C^2} \wedge \frac{(1-s)\gamma}{s}.$$

$A(C, s)$ is a dense open set! May have small Lebesgue measure.

Porous sets

Definition

The set $H \subset \mathbb{R}$ is porous at x if

$$\limsup_{r \to 0} \frac{\text{length of the largest subinterval of } (x - r, x + r) \setminus H}{2r} > 0.$$

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- For $H = [0, \infty) \setminus A(C, s)$ the set of bad time points

$$\mathbf{P}(H \text{ is porous at } 1) = 1 \quad \implies \quad \forall t > 0,\ \mathbf{P}(H \text{ is porous at } t) = 1$$

$$\implies \quad \mathbf{P}(H \text{ is porous at a.e. } t > 0) = 1 \quad \implies \quad \mathbf{P}(\lambda(H) = 0) = 1$$

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New goal:
The set of good time points contains sufficiently large intervals near 1.

$\limsup_{n\to\infty} \frac{\min_{0\le k<n} |\beta^{(k)}_{\gamma^*_n}|}{\sqrt{1-\gamma^*_n}} > 0$ a.s. is enough for strong mixing

- Here $\gamma_n = \sup\left\{t \le 1 \,:\, \beta^{(n)}_t = 0\right\}$, $\gamma^*_n = \max_{0\le k\le n} \gamma_k$.

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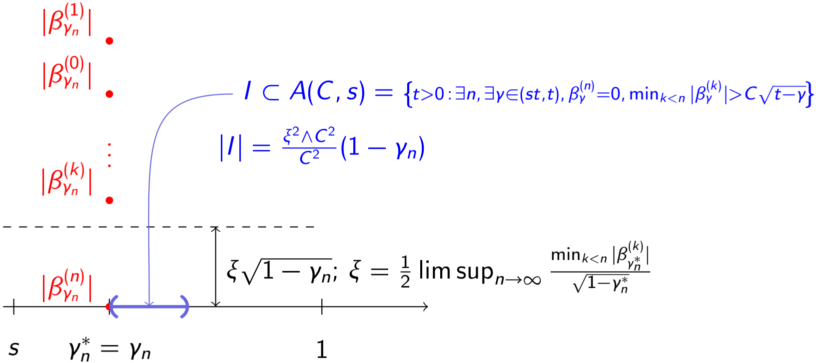
- By Malric's density theorem $\gamma_n^* \to 1$.



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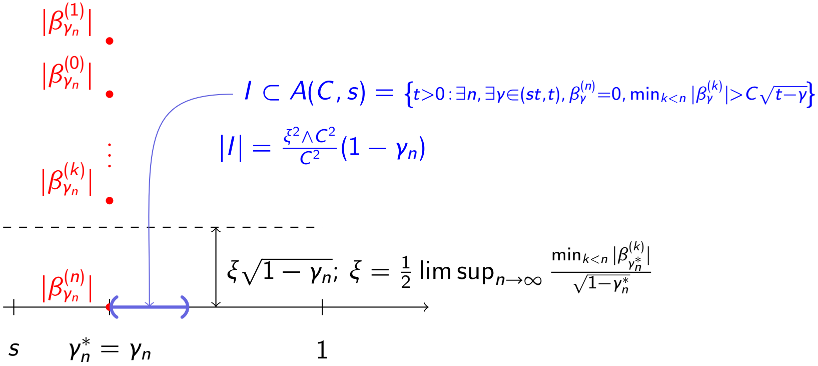

- $I \subset A(C,s)$, the length of I is proportional to $(1-\gamma_n) = \delta'$.
- $A(C,s)$ is of full Lebesgue measure for all C, s, etc..

$\liminf_{n\to\infty} \frac{Z_{n+1}}{Z_n} < 1$ a.s. also guarantees strong mixing

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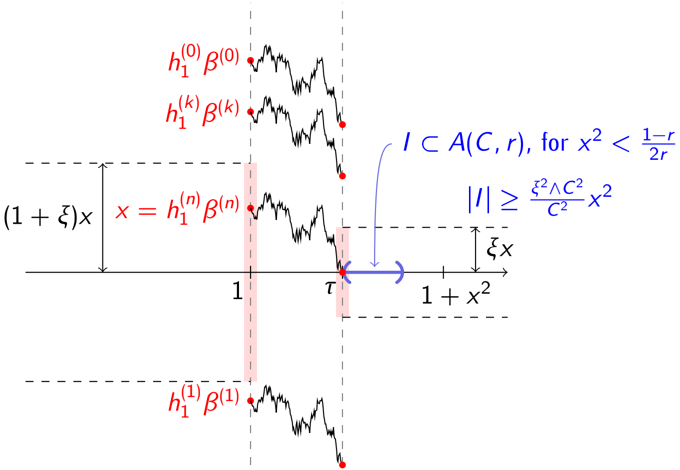
- This condition is obtained similarly, by considering the right neighborhood of 1.



Remark on $X = \liminf \frac{Z_{n+1}}{Z_n}$ and $Y = \limsup \frac{\min_{0\le k<n} |\beta^{(k)}_{\gamma^*_n}|}{\sqrt{1-\gamma^*_n}}$

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Working a bit harder, one can obtain that both X and Y are invariant, and

- Either $Y = 0$ a.s.,
- or $0 < \mathbf{P}(Y = 0) < 1$ and $\mathbf{T}$ is not ergodic,
- or $Y > 0$ a.s. and then $Y = \infty$ and $\mathbf{T}$ is strongly mixing.

Also

- Either $X = 1$,
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Remark: There is a hope that $\mathbf{P}(X = 1) = 1$ is impossible. Then

- $\mathbf{P}(X = 1) > 0 \implies X$ is not constant, hence X is a nontrivial invariant variable.
- Both X, and Y characterize ergodicity: $X < 1 \Leftrightarrow Y > 0 \Leftrightarrow \mathbf{T}$ is strongly mixing.

$\liminf_{x\searrow 0} \frac{|\beta_1^{(\nu(x))}|}{x} < 1 \Leftrightarrow X = \liminf_{n\to\infty} \frac{Z_{n+1}}{Z_n}$

- Here $\nu(x) = \inf\left\{ n \geq 0 \,:\, |\beta_1^{(n)}| < x \right\}$ and $Z_n = \min_{k\leq 0<n} |\beta_1^{(k)}|$.
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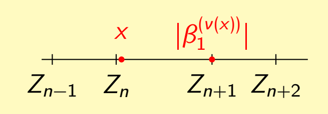

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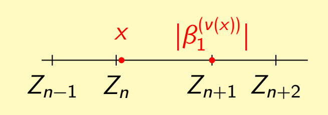

- Claim. $\{x\nu(x) \; : \; x \in (0,1)\}$ is tight $\implies$ $X = \liminf_{x \searrow 0} \frac{|\beta_1^{(\nu(x))}|}{x} < 1$ a.s. $\implies$ $\mathbf{T}$ is strongly mixing.
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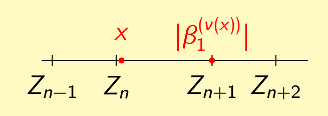

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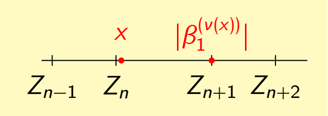

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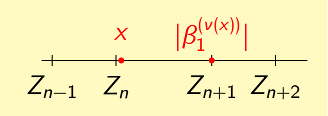

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&\leq \sup_{x\in(0,1)} \mathbf{P}\left(x\nu(x) > K\right) + (1+K)\delta.
\end{aligned}$$

$$\mathbf{P}\left(X = 1\right) \leq \inf_K \inf_\delta \mathbf{P}\left(x\nu(x) > K\right) + (1+K)\delta.$$

Is $\{x\nu(x) : x > 0\}$ tight?

Recall that
$\sup_{x\in(0,1)} \mathbf{E}\,(x\nu(x)) < \infty \implies \{x\nu(x) : x \in (0,1)\}$ is thight (by Markov inequality) $\implies$ **T** is strongly mixing.

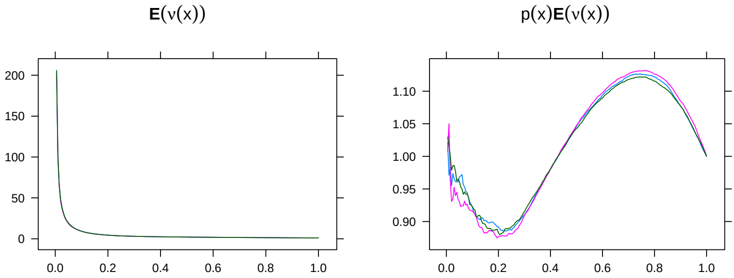

Figure: $\mathbf{E}\,(\nu^*(x))$ estimated from long runs of a SRW (number of iteration: 10^5, number of steps of SRW: 10^9).
On the x-axis the probability $p(x) = \mathbf{P}\left(|\beta_1^{(0)}| < x\right)$ is given.

Density of $\frac{1}{x}|\beta_1^{(\nu(x))}|$

- Consider the natural extension of $(\Omega, \mathcal{B}, \mathbf{P}, T)$. Then T is an invertible measure preserving transformation on the extension.
  That is
  - $\Omega = \mathbb{C}[0, \infty)^{\mathbb{Z}}$,
  - for $\omega = (\omega_n)_{n \in \mathbb{Z}}$ $(T\omega)_n = \omega_{n+1}$ and $\beta^{(n)}(\omega) = \omega_n$,
  - $\mathbf{P}$ is such that $\beta^{(k)}, \beta^{(k+1)}, \dots$ has the same joint law as $(\beta, \mathbf{T}^1\beta, \dots)$ for all $k \in \mathbb{Z}$.

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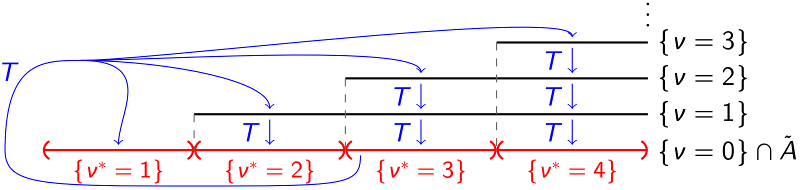

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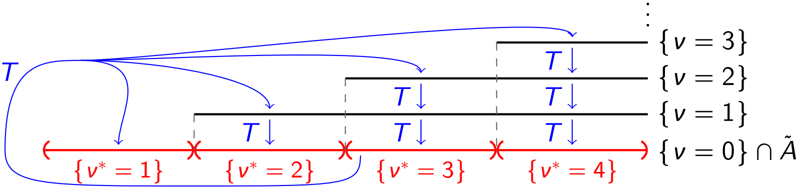
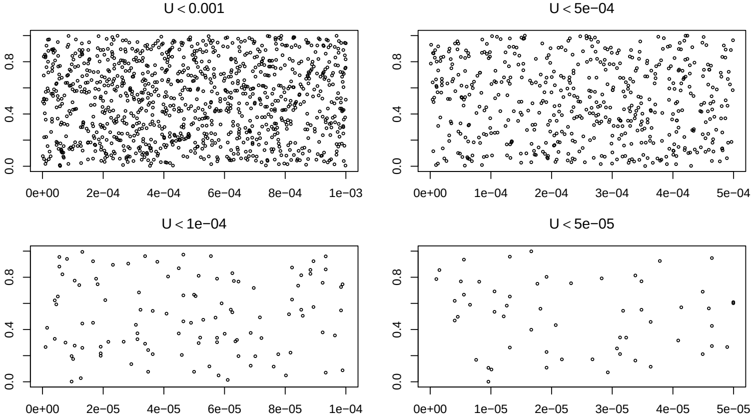


- The density f_x of $\frac{1}{x}|\beta_1^{(\nu(x))}|$ is obtained by conditioning

$$f_x(y) = 2\varphi(yx)\mathbf{E}\left(x\nu^*(x)\,\middle|\, |\beta_1^{(0)}| = yx\right), \quad \text{for } y \in (0,1)$$

The density $\mathbf{E}\left(x\nu^*(x) \,\middle|\, |\beta_1^{(0)}| = yx \right)$?



Joint behaviour of $|\beta_1^{(0)}|$ and $\nu^*(x)$ given $|\beta_1^{(0)}| < x$. Both are rescaled to uniform variables.

$\frac{1}{x}|\beta_1^{(0)}|$ seems to be conditionally independent of $x\nu^*(x)$,

(From one long random walk: number of steps 10^{13}, number of iterated paths 10^6.)

$\lim_{x\to 0^+} \mathbf{E}\left(x\nu^*(x) \,\middle|\, |\beta_1^{(0)}| = yx \right) =?$

- Conjecture: $\frac{1}{x}|\beta_1^{(\nu(x))}|$ converges in distribution to a uniform variable. Actually the density seems to go to 1 as $x \to 0^+$.
- Playing with two types of expected return times one can show that

$$\liminf_{x\to 0^+} \mathbf{P}\left(|\beta_1^{(\nu(x))}| < x/2 \right) > 0.$$

- This is enough

$$\liminf_{x\to 0^+} \frac{|\beta_1^{(\nu(x))}|}{x} < 1 \quad \text{with positive probability.}$$

- Recall that then both

$$X = \liminf_{x\to 0^+} \frac{\min_{0\le k\le n} |\beta_1^{(k)}|}{\min_{0\le k< n} |\beta_1^{(k)}|}, \qquad Y = \limsup_{x\to 0^+} \frac{\min_{0\le k< n} |\beta_{\gamma_n^*}^{(k)}|}{\sqrt{1-\gamma_n^*}}$$

characterize ergodicity: $X < 1 \Leftrightarrow Y > 0 \Leftrightarrow \mathbf{T}$ is strongly mixing $\Leftrightarrow \mathbf{T}$ is ergodic.

Conclusion

- Marc Malric has proved that the orbit of a typical sample path meets every open set.
- To prove strong mixing only certain open sets has to be considered.
- For these open sets

  • Tightness of the family rescaled hitting times would be enough.

  • or a quantitative result is needed: the expected hitting times do not growth faster than the inverse of the size of these open sets.

Thank you for your attention!

Happy birthday!