TRANSPORT INEQUALITIES FOR STOCHASTIC PROCESSES

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INTRODUCTION

Three problems about rank-based processes

The Atlas model

- Define ranks: $x_{(1)} \geq x_{(2)} \ldots \geq x_{(n)}$. Fix $\delta > 0$.
- SDE in $\mathbb{R}^n$:

$$X_i(t) = x_0 + \delta \int_0^t 1\{X_i(s) = X_{(n)}(s)\}\, ds + W_i(t), \quad \forall\, i.$$

- The market weight: $S_i(t) = \exp(X_i(t))$,

$$\mu_i(t) = \frac{S_i}{S_1 + S_2 + \ldots + S_n}(t).$$

- Banner, Fernholz, Karatzas, P.- (Pitman, Chatterjee), Shkolnikov, Ichiba and several more.

A CURIOUS SHAPE

Power law decay of real market weights with rank:

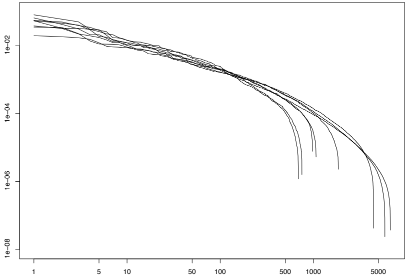

Figure 1: Capital distribution curves: 1929–1999

- $\log \mu_{(i)}$ vs. $\log i$.
- Dec 31, 1929 - 1999.
- Includes all NYSE, AMEX, and NASDAQ.

Problem 1

- How to show concentration of the shape of market weights?
- Fix $J \ll N$. Linear regression through

$$(\log i,\ \log \mu_{(i)}(t)), \quad 1 \leq i \leq J.$$

- Slope $-\alpha(t)$.
- Estimate fluctuation of the process $\{\alpha(s),\ 0 \leq s \leq T\}$.

Problem 2

- Lipschitz functions $F_1(T, B(T)), \ldots, F_d(T, B(T))$.
- Define
$$M_i(t) := E\left[F_i(T, B(T)) \mid B(t)\right].$$
- Suppose
$$P\left(\sup_i M_i(t) \leq a(t),\ 0 \leq t \leq T\right) \geq 1/2.$$
- What is
$$P\left(\sup_i M_i(t) > a(t) + \alpha\sqrt{t},\ 1 \leq t \leq T \ \middle|\ \sup_i M_i(1) > a(1)\right)?$$

Problem 3

- Back to rank-based models.
- Suppose $V^{\pi}(t)$ wealth ($V^{\pi}(0) = 1$)- portfolio π.
- $\pi = \mu$ - market portfolio.
- How does V^{π} compare with V^{μ} ?

$$P\left(V^{\pi}(t)/V^{\mu}(t) \geq a(t)\right) \leq \exp\left(-r(t)\right),$$

explicit $a(t)$ and $r(t)$.

And now the answers ...

Problem 1: fluctuation of slope

Theorem (P.-'10, P.-Shkolnikov '10)

Suppose market weights are running at equilibrium.
Take $T = N/\delta^2$.

Let $\bar{\alpha} = \sup_{0 \le s \le T} \alpha(s)$.

$$P\left(\bar{\alpha} > m_\alpha + r\sqrt{N}\right) \le 2\exp\left(-\frac{r^2}{2\sigma^2}\right).$$

Here m_α *=median and* $\sigma^2 = \sigma^2(\delta, J)$.

PROBLEM 2: BAD PRICES

THEOREM (P. '12)

For some absolute constant $C > 0$:

$$P\left(\sup_i M_i(t) > a(t) + \alpha\sqrt{t},\ 1 \le t \le T \ \middle|\ \sup_i M_i(1) > a(1)\right) \approx C T^{-\alpha^2/8}.$$

Compare with square-root boundary crossing.

Problem 3: Performance of portfolios

- Symmetric functionally generated portfolio G.
- π depends only on market weights.
- Market, diversity-weighted, entropy-weighted portfolios.

Theorem (Ichiba-P.-Shkolnikov '11)

Let $R(t) = V^{\pi}(t)/V^{\mu}(t)$*.*

$$P\left(R(t) \geq c^{+} G(\mu(t))/G(\mu(0))\right) \leq \exp\left[-\alpha^{+} t\right]$$
$$P\left(R(t) \leq c^{-} G(\mu(t))/G(\mu(0))\right) \leq \exp\left[-\alpha^{-} t\right].$$

Here $c^{\pm}, \alpha^{\pm}$ *explicit.*

Transportation - Entropy - Information Inequalities

Transportation Inequalities

TCI (Ω, d) - metric space. P, Q - prob measures.

$$\mathcal{W}_p(Q, P) = \inf_{\pi} \left[E d^p(X, X') \right]^{1/p} .$$

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- P satisfies T_p if $\exists\ C > 0$:

$$\mathcal{W}_p(Q,P) \leq \sqrt{2CH(Q \mid P)}.$$

- $H(Q \mid P) = E_Q \log(dQ/dP)$ or ∞.

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- Related: Bobkov and Götze, Bobkov-Gentil-Ledoux, Dembo, Gozlan-Roberto-Samson, Otto and Villani, Talagrand.

Marton's argument

- $T_p, \ p \geq 1 \Rightarrow$ Gaussian concentration of Lipschitz functions.
- If $f : \Omega \to \mathbb{R}$ - Lipschitz.

$$|f(x) - f(y)| \leq \sigma d(x, y).$$

- Then f has Gaussian tails:

$$P\left(|f - m_f| > r\right) \leq 2e^{-r^2/2C\sigma^2}.$$

- Fix $p = 2$ from now on.

Idea of proof for Problem 1

THE WIENER MEASURE

- Consider $\Omega = C[0,T]$, $\mathbf{d}(\omega, \omega') = \sup_{0 \le s \le T} |\omega(s) - \omega'(s)|$.
- (Feyel-Üstünel '04, P. '10)

$$P = \text{Wiener measure satisfies } T_2 \text{ with } C = T.$$

- Related: Djellout-Guillin-Wu, Fang-Shao, Fang-Wang-Wu, Wu-Zhang.

Proof

- Proof: If $Q \ll P$, then by Girsanov

$$d\omega(t) = b(t,\omega)dt + d\beta(t).$$

  Here $\beta \sim P$.

- $\mathcal{W}_2(Q,P) \leq \left[E_Q d^2(\omega,\beta)\right]^{1/2} \leq \sqrt{2TH(Q \mid P)}$.

Examples

- How to show local time at zero has Gaussian tail?
- $L_0(T)$ is not Lipschitz w.r.t uniform norm.
- Lévy representation:

$$L_0(T) = -\inf_{0 \le s \le t} \beta(s) \wedge 0.$$

- Lipschitz function of the entire path. Thus

$$P\left(|L_0(T) - m_T| > r\right) \le 2e^{-r^2/2T}.$$

BM in $\mathbb{R}^n$

- Multidimensional Wiener measure satisfies T_2.
- Uniform metric

$$\mathbf{d}^2(\omega, \omega') = \frac{1}{n} \sum_{i=1}^{n} \sup_{0 \le s \le T} \left| \omega(s) - \omega'(s) \right|^2 .$$

- Skorokhod map

$$\mathcal{S} : \text{BM in } \mathbb{R}^n \mapsto \text{RBM in polyhedra.}$$

- Deterministic map. Rather abstract and complicated.
- But Lipschitz.

The Lipschitz constant

Theorem (P. - Shkolnikov '10)

The Lipschitz constant of $\mathcal{S}$ is $\leq 2n^{5/2}$.

- The slope $\alpha(t)$ is a linear map.

  BM on $\mathbb{R}^n \rightarrow$ RBM on wedge $\rightarrow$ slope of regression.

- Evaluate Lipschitz constant. Estimate concentration.

Idea of proof for Problem 2

A DIFFERENT METRIC

- For $\omega, \omega' \in C^d[0,\infty)$:

$$\sigma_r = \inf\{t \geq 0 : \ \sigma_r(\omega,\omega') > r\}.$$

- Consider $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$

$$\Phi_1 := \left\{\varphi \geq 0, \ \varphi \downarrow, \int_0^\infty \varphi^2(s)ds \leq 1\right\}.$$

- (P. '12) A metric on paths:

$$\rho(\omega,\omega') := \left[\sup_{\varphi\in\Phi_1} \int_0^\infty \varphi\left(\sigma_r\right) dr\right]^{1/2}.$$

Generalized TCI

Theorem (P. '12)

P - multidimension Wiener measure.

$$\mathcal{W}_2(Q, P) \le \sqrt[4]{2H(Q \mid P)}.$$

With respect to ρ.

An example

- P-Wiener measure. Two event processes: $1 \leq t \leq T$.

$$A_T = \{\beta(s) \leq \sqrt{s},\ 1 \leq s \leq T\} \quad B_T = \{\beta(s) \geq 2\sqrt{s},\ 1 \leq s \leq T\}.$$

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$$Q_1 = P(\cdot \mid A_T), \quad Q_2 = P(\cdot \mid B_T).$$

- Couple $(X, Y) \sim (Q_1, Q_2)$.

$$\sigma_{\sqrt{s}}(X, Y) \le s, \quad 1 \le s \le T.$$

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- $\varphi \downarrow$ and ≥ 0:

$$\int_1^{\sqrt{T}} \varphi(\sigma_r) dr = \int_1^T \varphi(\sigma_{\sqrt{s}}) \frac{ds}{2\sqrt{s}} \geq \int_1^T \varphi(s) \frac{ds}{2\sqrt{s}}.$$

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- Take

$$\varphi(s) = \frac{2}{\sqrt{\log T}} \frac{1}{2\sqrt{s}} 1\{1 \le s \le T\}.$$

EXAMPLE CONTD.

- Thus
$$\mathcal{W}_2^2(Q_1, Q_2) \geq \frac{1}{2}\sqrt{\log T}.$$

Example contd.

- Thus
$$\mathcal{W}_2^2(Q_1, Q_2) \geq \frac{1}{2}\sqrt{\log T}.$$
- $\frac{1}{\sqrt{2}}\sqrt[4]{\log T} \leq \mathcal{W}_2(Q_1, Q_2) \leq \mathcal{W}_2(Q_1, P) + \mathcal{W}_2(Q_2, P)$

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  $\leq \sqrt[4]{2H(Q_1 \mid P)} + \sqrt[4]{2H(Q_2 \mid P)}$

Example contd.

- Thus
$$\mathcal{W}_2^2(Q_1, Q_2) \geq \frac{1}{2}\sqrt{\log T}.$$
- $\frac{1}{\sqrt{2}}\sqrt[4]{\log T} \leq \mathcal{W}_2(Q_1, Q_2) \leq \mathcal{W}_2(Q_1, P) + \mathcal{W}_2(Q_2, P)$

$$\leq \sqrt[4]{2H(Q_1 \mid P)} + \sqrt[4]{2H(Q_2 \mid P)}$$

$$\leq \sqrt[4]{2\log\frac{1}{P(A_T)}} + \sqrt[4]{2\log\frac{1}{P(B_T)}}.$$

Idea of proof for problem 3.

Transportation-Information inequality

- $\mathcal{E}$ - Dirichlet form. Fisher Information:

$$I(\nu \mid \mu) := \mathcal{E}(\sqrt{f}, \sqrt{f}), \quad \text{if } d\nu = f d\mu.$$

- μ satisfies $\mathcal{W}_1 I(c)$ inequality if

$$\mathcal{W}_1^2(\nu, \mu) \leq 4c^2 I(\nu \mid \mu), \quad \forall\, \nu.$$

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$$\mathcal{W}_1^2(\nu, \mu) \leq 4c^2 I(\nu \mid \mu), \quad \forall \nu.$$

- Allows precise control of additive functionals.

Theorem (Guillin et al.)

Consider

$$\mathcal{W}_1(\nu, \mu) = \|\nu - \mu\|_{\mathrm{TV}} .$$

$(X_t,\ t \geq 0)$ *Markov - invariant distribution* μ*.*

Suppose μ *- Poincaré ineq. Then* $\mathcal{W}_1 I$ *holds.*

Poincaré inequalities

Theorem (Guillin et al.)

Consider

$$\mathcal{W}_1(\nu,\mu) = \|\nu - \mu\|_{\mathrm{TV}}.$$

$(X_t,\ t \geq 0)$ *Markov - invariant distribution* μ.

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Gaps of rank-based processes stationary. Poincaré ineq holds.

Thank you Ioannis. Happy birthday.