Portfolio optimisation with small transaction costs
— an engineer's approach

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based on joint work with
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dedicated to Ioannis Karatzas

Outline

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Portfolio choice under transaction costs

using duality

- Cvitanić & Karatzas (1996):

formation to a hedging problem; the optimal portfolio is the one that hedges the inverse of marginal utility, evaluated at the shadow state-price density which solves the corresponding dual problem. This hedging-duality approach has been used previously in models of incomplete markets, markets with constraints, and markets with nonlinear drifts in the wealth process of the investor (Cvitanić and Karatzas 1992, 1993), but it seems to be new in the context of models with transaction costs. A related approach based on the stochastic max-

- Loewenstein (2000):

(1995). In any case, we can construct an economy without transactions costs which supports the optimal trading strategies and solution to the transaction cost problem. This leads to a characterization of the optimal solution as the least favorable of a specific set of economies with no transactions costs.

Shadow price processes in optimisation

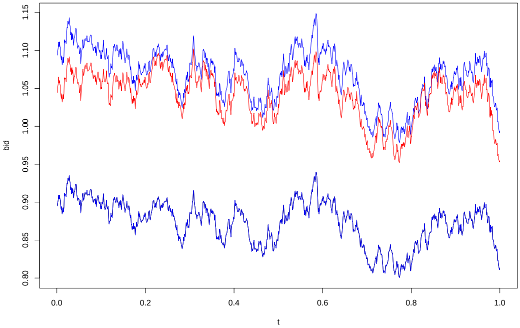
The basic principle

- Goal:

$$\max_{\varphi} E(u(V_T(\varphi)))$$

  $V(\varphi)$ wealth under transaction costs

- Ex. $\widetilde{S}$ in $[\underline{S}, \overline{S}]$ such that
  - optimiser $\varphi^\star$ maximises $\varphi \mapsto E(u(v_0 + \varphi \bullet \widetilde{S}_T))$,
  - $V_T(\varphi^\star) = v_0 + \varphi^\star \bullet \widetilde{S}_T$.



What about explicit results —

using shadow prices?

- Cvitanić & Karatzas (1996):

> (1994), is suggested in Cadenillas and Haussman (1993). The typical approach to utility maximization under transaction costs has been the analytical study of the value function and the description of the optimal strategy as one with no transactions in a certain region, but with minimal transactions at the boundary in order always to keep the holdings vector inside the region. Such was the spirit of the pioneering work of Magill and Constantinides

> Panas, and Zariphopoulou (1993). Our approach gives different insights and can be applied to the case of time-dependent and random market coefficients, but it provides no explicit description of optimal strategies, except for the cases in which it is optimal to not trade at all. The latter is the case when the difference between the return rate of the stock and

> it is. It should be of considerable interest to find additional examples that admit explicit solutions.

- but cf. K. & Muhle-Karbe (2009), Kühn & Stroh (2010), Gerhold et al. (2011), Herczegh & Prokaj (2011), Choi et al. (2012)

What about explicit results —

using shadow prices?

- Problem: explicit results are very rare.
  (e.g. Davis & Norman 1990, ...)
- Way out: look at asymptotics for small transaction costs.
  (Constantinides 1986, Whalley & Wilmott 1997, Atkinson & Al-Ali 1997, Janeček & Shreve 2004, Rogers 2004, ...)

Outline

Our setup

The portfolio optimisation problem

- Want to maximise

$$\varphi \mapsto E(-\exp(-pV_T(\varphi)))$$

- $V(\varphi)$ wealth process under transaction costs
- bid price $\underline{S} = (1-\varepsilon)S$, ask price $\overline{S} = (1+\varepsilon)S$
- Why exponential utility $u(x) = -\exp(-px)$?
  - It allows for random endowment (= hedging problem) by a measure change.
  - Solution does not depend on initial wealth.
  - Utility on $\mathbb{R}$ better suited for hedging than utilities on $\mathbb{R}_+$.
  - Exponential utility often leads to simple structure.

Our setup

The traded asset

- Traded asset

$$dS_t = b_t dt + \sigma_t dW_t$$

  - univariate
  - continuous
  - otherwise rather arbitrary
- Frictionless optimiser φ is assumed to be known.

Shadow price approach

Look for

- optimal strategy φ^ε,
- shadow price process $\widetilde{S}^\varepsilon$,
- dual martingale Z^ε,

which satisfy

- $Z_T = u'(v_0 + \varphi^\varepsilon \bullet \widetilde{S}^\varepsilon_T)$,
- Z^ε martingale,
- $Z^\varepsilon \widetilde{S}^\varepsilon$ martingale,
- φ^ε changes only when $\widetilde{S}^\varepsilon_t \in \{\underline{S}_t, \overline{S}_t\}$.

Optimality of φ^ε

from an engineer's perspective

For any competitor ψ we have

$$
\begin{aligned}
E(u(V_T(\psi))) &\leq E(u(v_0 + \psi \bullet \widetilde{S}^\varepsilon_T)) \\
&\leq E(u(v_0 + \varphi^\varepsilon \bullet \widetilde{S}^\varepsilon_T)) + E\Big(u'(v_0 + \varphi^\varepsilon \bullet \widetilde{S}^\varepsilon_T)((\psi - \varphi^\varepsilon) \bullet \widetilde{S}^\varepsilon_T)\Big) \\
&= E(u(V_T(\varphi^\varepsilon))) + E\Big(Z_T((\psi - \varphi^\varepsilon) \bullet \widetilde{S}^\varepsilon_T)\Big) \\
&= E(u(V_T(\varphi^\varepsilon))).
\end{aligned}
$$

(Second inequality follows from $u(y) \leq u(x) + u'(x)(y - x)$.)

Approximate solution

Look for

- approximately optimal strategy φ^ε,
- approximate shadow price process $\widetilde{S}^\varepsilon$,
- approximate dual martingale Z^ε,

which satisfy

- $Z_T = u'(v_0 + \varphi^\varepsilon \bullet \widetilde{S}^\varepsilon_T) + O(\varepsilon)$,
- Z^ε has drift $o(\varepsilon^{2/3})$,
- $Z^\varepsilon \widetilde{S}^\varepsilon$ has drift $O(\varepsilon^{2/3})$,
- φ^ε changes only when $\widetilde{S}^\varepsilon_t \in \{\underline{S}_t, \overline{S}_t\}$.

Approximate optimality of φ^ε

from an engineer's perspective

For any competitor ψ^ε with $\psi^\varepsilon - \varphi = o(1)$ we have

$$
\begin{aligned}
&E(u(V_T(\psi^\varepsilon))) \leq E(u(v_0 + \psi^\varepsilon \bullet \widetilde{S}^\varepsilon_T)) \\
&\quad \leq \quad E(u(v_0 + \varphi^\varepsilon \bullet \widetilde{S}^\varepsilon_T)) + E\Big(u'(v_0 + \varphi^\varepsilon \bullet \widetilde{S}^\varepsilon_T)((\psi^\varepsilon - \varphi^\varepsilon) \bullet \widetilde{S}^\varepsilon_T)\Big) \\
&\quad = \quad E(u(V_T(\varphi^\varepsilon))) + E\Big(Z_T((\psi^\varepsilon - \varphi^\varepsilon) \bullet \widetilde{S}^\varepsilon_T)\Big) + o(\varepsilon^{2/3}) \\
&\quad = \quad E(u(V_T(\varphi^\varepsilon))) + o(\varepsilon^{2/3}).
\end{aligned}
$$

Compare with

$$E(u(V_T(\varphi^\varepsilon))) - E(u(v_0 + \varphi \bullet \widetilde{S}_T)) = O(\varepsilon^{2/3}).$$

How to find an approximate solution

The ansatz

- How to find φ^ε, $\widetilde{S}^\varepsilon$, Z^ε, no-trade bounds $\Delta\varphi^\pm$?
- Write

$$\begin{aligned} \varphi^\varepsilon &= \varphi + \Delta\varphi, \\ \widetilde{S}^\varepsilon &= S + \Delta S, \\ Z^\varepsilon &= Z(1 + K), \end{aligned}$$

  where φ, Z are optimal for the frictionless problem.
- Ansatz:

$$\begin{aligned} \Delta S &= f(\Delta\varphi, \dots), \\ f(x) &= \alpha x^3 - \gamma x \end{aligned}$$

How to find an approximate solution

The solution

- This works for

$$\Delta S_t = \left((\gamma_t \Delta\varphi)^3 - 3\gamma_t x\right)\frac{\varepsilon S_t}{2}$$

with

$$\gamma_t = \sqrt[3]{\frac{2p}{3\varepsilon S_t}\frac{c_t^S}{c_t^\varphi}},$$

where $d\langle S,S\rangle_t = c_t^S dt$ and $d\langle \varphi,\varphi\rangle_t = c_t^\varphi dt$.

- no-trade bounds

$$\boxed{\Delta\varphi_t^\pm = \pm\frac{1}{\gamma_t} = \pm\sqrt[3]{\frac{3\varepsilon S_t}{2p}\frac{c_t^\varphi}{c_t^S}}}$$

- certainty equivalent of utility loss

$$\boxed{-E\left(Z_T\int_0^T \frac{p}{2}(\Delta\varphi_t^\pm)^2 d\langle S,S\rangle_T\right)}$$

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Outline

The multivariate case

The setup

- Traded asset
$$dS_t = b_t dt + \sigma_t dW_t$$
with $\mathbb{R}^d$-valued S and multivariate Wiener process W
(e.g. Akian et al. 1996, Liu 2004, Lynch & Tan 2006, Muthuraman & Kumar 2006, Atkinson & Ingpochai 2007, Goodman & Ostrov 2007, Law et al. 2009, Bichuch & Shreve 2012)
- Frictionless optimiser φ is assumed to be known.

The multivariate case

The ansatz

- How to find φ^ε, $\widetilde{S}^\varepsilon$, Z^ε, no-trade region?
- Write

$$\begin{aligned} \varphi^\varepsilon &= \varphi + \Delta\varphi, \\ \widetilde{S}^\varepsilon &= S + \Delta S, \\ Z^\varepsilon &= Z(1+K), \end{aligned}$$

  where φ, Z are optimal for frictionless problem.
- Ansatz:

$$\begin{aligned} \Delta S^i &= f_i(\Delta\varphi, \dots), \quad i = 1, \dots, d \\ f_i(x) &= \left((\gamma_i^\top x)^3 - 3\gamma_i^\top x\right) \frac{\varepsilon S^i}{2} \end{aligned}$$

  No-trade region is parallelotop spanned by d vectors $\pm a_1 \dots, \pm a_d \in \mathbb{R}^d$.

The multivariate case

The solution

- $$\Delta S^i = \left((\gamma_i^\top x)^3 - 3\gamma_i^\top x\right)\frac{\varepsilon S^i}{2}$$

  works for $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{R}^{d\times d}$ given by

  $$\gamma_{ij} = \sqrt[3]{\frac{2p}{3\varepsilon S^i}\frac{1}{(c^S c^\varphi c^S)_{ii}}} c^S_{ij},$$

  where $d\langle S^i, S^j\rangle_t = (c^S_{ij})_t dt$ and $d\langle \varphi^i, \varphi^j\rangle_t = (c^\varphi_{ij})_t dt$.
- no-trade parallelotop is spanned by $a_1, \ldots, a_d \in \mathbb{R}^d$, where $a = (a_1, \ldots, a_d) \in \mathbb{R}^{d\times d}$ is given by

  $$\boxed{a_{ij} = (\gamma^{-1})_{ij} = \sqrt[3]{\frac{3\varepsilon S^i}{2p}(c^S c^\varphi c^S)_{jj}}(c^S)^{-1}_{ij}}$$
- certainty equivalent of utility loss more complicated than in univariate case

The multivariate case

The solution

- $$\Delta S^i = \left((\gamma_i^\top x)^3 - 3\gamma_i^\top x\right)\frac{\varepsilon S^i}{2}$$
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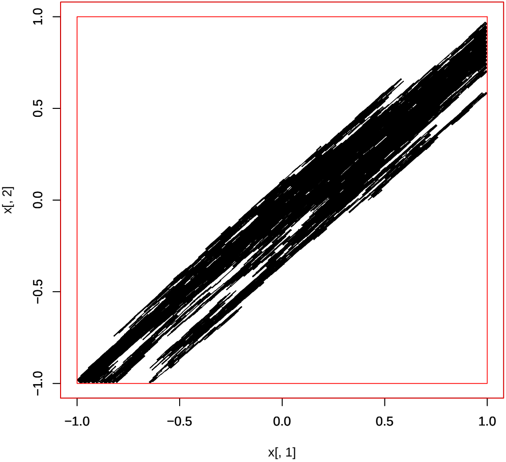
Ingredient of the certainty equivalent of utility loss

Covariance matrix of correlated Brownian motion with oblique reflection at the boundary



What is the covariance matrix of the stationary law?

Conclusion

- Shadow price approach is useful to the engineer as well.
- One can get quite far with regards to explicit formulas for asyptotically optimal portfolios.
- Rigorous theorems still left to future research.