Lect 6

Here we will focus on limit shapes for the stochastic 6-vertex model.

Boundary distribution

- $b(\sigma_1, \ldots, \sigma_N) \geq 0$, $\sigma_i = +$ (no path), $-$ (path)
- $\sum_{\{\sigma\}} b(\sigma) = 1$

The partition function for a cylinder with boundary distribution b at the top end and b' at the lower end.

$$\langle b, Z_{NM} b' \rangle = \sum_{\sigma, \sigma'} b(\sigma) Z_{NM}(\sigma, \sigma') b'(\sigma')$$

Boundary state is <u>thermodynamically relevant</u> if as $\varepsilon \to 0$, $N = L/\varepsilon$

$$b(\{\sigma\}) \to \exp\left(-\frac{1}{\varepsilon^2}\beta(\partial_x X)\right)$$

Here we assume

$\{\sigma_i\}_{i=1}^{N} \mapsto$ density of paths $\partial_x X(x)$
(p.w. continuous)

$\beta(\partial_x X)$ does not have to be linear or local in $\partial_x X$

<u>Example 1</u> uniform distribution

$$\beta(\partial_x \varphi) = 0$$

<u>Example 2</u>: Let $\tau_1, \dots, \tau_n = \pm$

Define

$$b(\sigma) = Z_{N,K}(\sigma, \tau)$$

Assuming $\{\tau_i\}_{i=1}^{N} \mapsto \partial_x \varphi_+(x)$

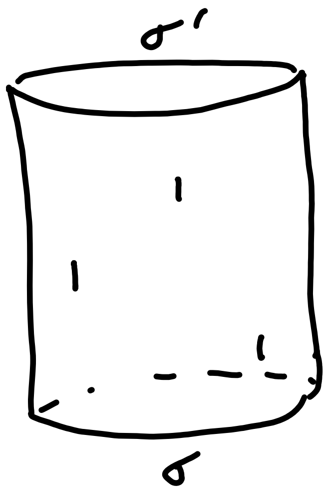
and that the limit shape conjecture for the 6-vertex model is valid, we have:

$$Z(\sigma) = \exp\left(\frac{1}{\varepsilon^2} S_u[\varphi_0] \left(1 + o(1)\right)\right)$$

where φ_0 is the minimizer of $S_u[\varphi]$ on the space $\mathcal{H}_{\chi, \varphi_+}$.

The dependence of $S_u[\varphi_0]$ on $\partial_x \chi$ is nonlocal and complicated.

Correlation functions



$e_1, \dots, e_n =$
$=$ vertical edges

$$p(e_1, \dots, e_n)_{\sigma,\sigma'}$$

$$= \mathbb{E}(\sigma_{e_1} \cdots \sigma_{e_n})_{\sigma,\sigma'}$$

probability that $e_1, \dots, e_n$ are covered by a configuration of paths

$$p(e_1 \dots e_n)_{\sigma\sigma'} = \frac{1}{Z(\sigma,\sigma')} \sum_{\{\tau_1\}} Z(\sigma', \tau_1)\, P_{e_1}(\tau_1)$$

$$\cdots Z(\tau_i, \tau_{i+1})\, P_{e_{i+1}}(\tau_{i+1}) \cdots P_{e_n}(\tau_n)\, Z(\tau_n, \sigma)$$

Horizontal edges require slightly longer discussion in terms of transfer-matrices.

Stochastic 6-vertex model

$$H = -V = \eta/2$$

$$t = (D^{-\eta})^{\otimes N} \cdot \mathrm{tr}_a (D_a^{\eta} R_{a1}(u) \cdots$$
$$\cdots D_a^{\eta} R_{aN}(u))$$

$$(\mathbb{C}^2)^{\otimes N} = \bigoplus_{n=0}^{N} \mathcal{H}^{(n)}$$

weight subspace, $m = \#(\text{paths})$

$$t\Big|_{\mathcal{H}^{(n)}} (1 + b_1^{N-n} b_2^{n})^{-1} = M^{(n)}$$

is Markov $\Rightarrow$ its coeigenvector corresponding to the maximal eigenvalue $\lambda^{(n)}_{max} = 1$ is the uniform distribution

But we already have $\lambda^{(n)}_{max}$ from the Bethe ansatz.

Strange identity:

From the Bethe ansatz we expect that maximal eigenvalue of the transfer-matrix in the subspace with n paths is

$$\Lambda^{(n)}_{max}(u|\{\lambda\}) = e^{\frac{N\eta}{2}} sh(u+\eta)^N \prod_{j=1}^{n} \frac{sh(u-i\alpha_j-\frac{\eta}{2})}{sh(u-i\alpha_j+\frac{\eta}{2})} e^{-\frac{\eta n}{2}}$$

$$+ e^{-N\eta/2} \prod_{k=1}^{N} sh(u-v_k) \prod_{j=1}^{n} \frac{sh(u-i\alpha_j+\frac{3\eta}{2})}{sh(u-i\alpha_j+\frac{\eta}{2})} e^{\frac{\eta n}{2}},$$

$$\left(\frac{sh(i\alpha_a+\eta/2)}{sh(i\alpha_a-\eta/2)}\right)^N = e^{-\eta H} \prod_{b\neq a}^{n} \frac{sh(i\alpha_a-i\alpha_b+\eta)}{sh(i\alpha_a-i\alpha_b-\eta)}$$

$$e^{ip(\alpha)} = \frac{sh(i\alpha+\frac{\eta}{2})}{sh(i\alpha-\frac{\eta}{2})}, \quad e^{i\Theta(\alpha)} = \frac{sh(i\alpha+\eta)}{sh(i\alpha-\eta)}$$

$$p(\alpha_j) = 2iH + \frac{\pi(n+1-2j)}{N} + \frac{1}{N}\sum_{k\neq j}^{n} \Theta(\alpha_j-\alpha_k)$$

From the Markov property we know that the maximal eigenvalue is 1. Thus, one should expect

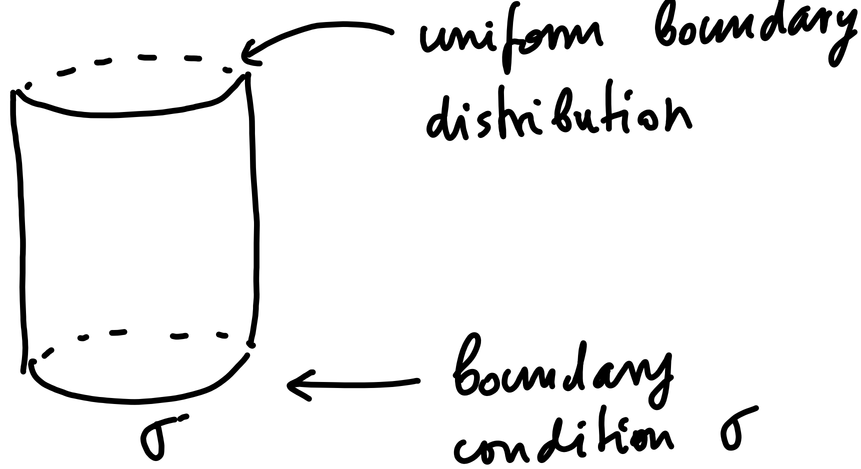
$$\Lambda^{(n)}_{max}(u|\{\lambda\}) = 1$$

Limit shapes for the stochastic 6-vertex model

The large deviation rate functional for the 6-vertex model on a cylinder in the presence of magnetic fields H and V

$$S[\varphi] = \int_0^L \int_0^T \left(\sigma(\partial_x \varphi, \partial_y \varphi) + V \partial_x \varphi + H \partial_y \varphi \right) dx\, dy$$

Consider the limit shape formation when



<u>Assume stochasticity:</u> $H = H_0 := \eta/2$

$V = V_0 := -\eta/2$

<u>The corresponding variational problem:</u>

Find minimizer of $S[\varphi]$ in the space of $\varphi(x,y)$, $x \in [0, L]$, $y \in [0, T]$

- $|\varphi(x,y) - \varphi(x',y)| < |x - x'|$

  $|\varphi(x,y) - \varphi(x,y')| < |y - y'|$

- free boundary condition at $y = T$

- $\varphi(x,0) = \varphi_0(x)$

$$\delta S_u'[\varphi] = \int_0^L \int_0^T \Big(\partial_1 \sigma(\varphi_x, \varphi_y)\, \delta\varphi_x +$$

$$+ \partial_2 \sigma(\varphi_x, \varphi_y)\, \delta\varphi_y + V_0\, \delta\varphi_x + H_0\, \delta\varphi_y \Big) dx\, dy$$

$$= - \int_0^L \int_0^T \big(\partial_x \sigma_1(\varphi_x, \varphi_y) + \partial_y \sigma_2(\varphi_x, \varphi_y) \big) dx\, dy$$

$$- \int_0^L \big(\sigma_2(\varphi_x, \varphi_y) - H_0 \big)\, \delta\varphi(x, T)\, dx$$

at $t = T$ we have:

$$\sigma_2(\varphi_x, \varphi_y) - H_0 = 0$$

<u>Proposition</u>. $\sigma_2(s,t) = \pm H_0$, $H_0 = \eta/2$ are exactly the critical curves for $\Delta > 1$. $\mathrm{Hess}(\sigma) = 0$ on these curves

<u>Proposition</u>. The differential equation

$$\sigma_2(\varphi_x, \varphi_y) = H_0 \qquad (*)$$

implies Euler-Lagrange equations and describes the minimizer of $S_u[\varphi]$.

∎

$$\partial_x \sigma_1 = \sigma_{11} \varphi_{xx} + \sigma_{12} \varphi_{xy} = \cdots$$

Since $\sigma_{11}\sigma_{22} - \sigma_{12}^2 = 0$

$$\cdots = \frac{\sigma_{12}^2}{\sigma_{22}} \varphi_{xx} + \sigma_{12}\varphi_{xy} = \frac{\sigma_{12}}{\sigma_{22}}\left(\sigma_{12}\varphi_{xx} + \sigma_{22}\varphi_{xy}\right) = \frac{\sigma_{12}}{\sigma_{22}} \partial_x \sigma_2 \underset{(*)}{\overset{\uparrow}{=}} 0$$

$\partial_y \sigma_2 = 0$ follows from $(*)$

$$\Rightarrow \quad \partial_x \sigma_1 + \partial_y \sigma_2 = 0$$

∎

Proposition On solutions to $(*)$ $S[\varphi]=0$

From the explicit description of the critical curve

$$\varphi_y = \frac{\varphi_x + th(u+\eta)}{1 + th(u+\eta)\,\varphi_x} \qquad \left(\begin{array}{c}\text{Sridhar, R.}\\ 2016\end{array}\right)$$

Differentiating in x:

$$(\star)\ \rho_y = -\frac{v^2-1}{(1+v\rho)^2}\,\rho_x\ , \qquad v = \frac{b_1-b_2}{2-b_1-b_2} = th(u+\eta)$$

(same as Borodin - Corwin - Gorin, 2014 for quadrant, different medod)

The limit to ASEP

The asymptotical expansion of (A) as

$$b_1 = \varepsilon p \,, \quad b_2 = \varepsilon q \,, \quad \varepsilon \to 0$$

gives

$$\rho_y = \rho_x - \frac{1}{2}\varepsilon (p-q) \rho \rho_x + O(\varepsilon^2)$$

- the zero order:

  $\rho_y = \rho_x$ is the continuum limit of the dynamics generated by the shift operator $t(0)$

- change of coordinates:

  $$\tilde{x} = x + y \,, \quad \tilde{y} = \varepsilon y$$

  $$\partial_x = \partial_{\tilde{x}} \,, \quad \partial_y = \varepsilon \partial_{\tilde{y}} + \partial_{\tilde{x}}$$

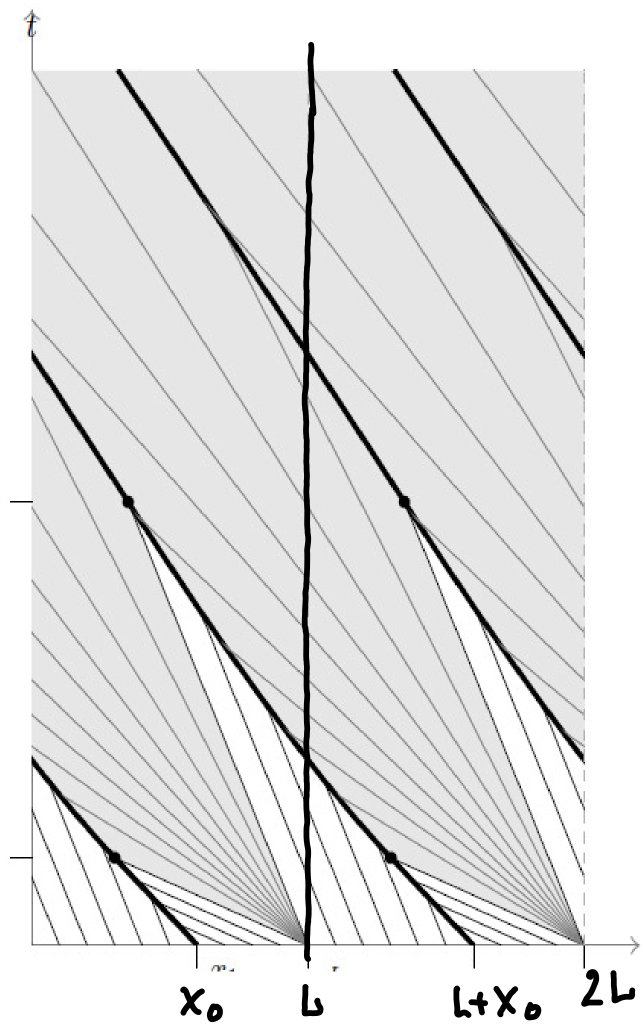
- the first order in $\tilde{x}, \tilde{y}$:

  $$\rho_{\tilde{y}} = \frac{1}{2}(p-q) \rho \rho_{\tilde{x}}$$

in Baxter's parametrization $\frac{1}{2}(p-q)=1$

$$\rho_{\tilde{y}} = \rho\,\rho_{\tilde{x}}$$

Burgers equation for limit shapes of ASEP.



Step initial conditions.

- Universality conjecture: on macroscopic scale fluctuations are universal (Gaussian Free Theory) in the universality class which includes dimer models and the 6-vertex model. They are determined by $S^{(2)}(\varphi_0)$.

- Higher finite size corrections: One should expect that leading terms of the log of Z are

$$\ln Z = \frac{1}{\varepsilon^2} S[\varphi_0] + \frac{1}{\varepsilon} B_1[\varphi_0]$$

$$+ B_0 \ln \varepsilon + \ln \det_{\zeta}(S^{(2)}) + C + o(1)$$

- The $\varepsilon \to 0$ is, in many ways similar to the semiclassical limit.