Lect 2

Note Title 12/1/2016

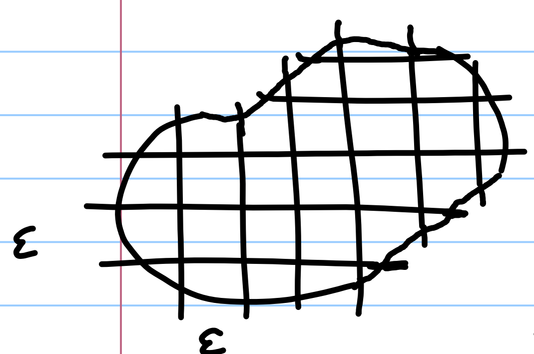
The thermodynamic limit

"$C \to \infty$"

- the asymptotic of Z_{C_ε} ?
- the asymptotic of local correlation fncns?
- the asymtotitic of the probability measure?

* * *

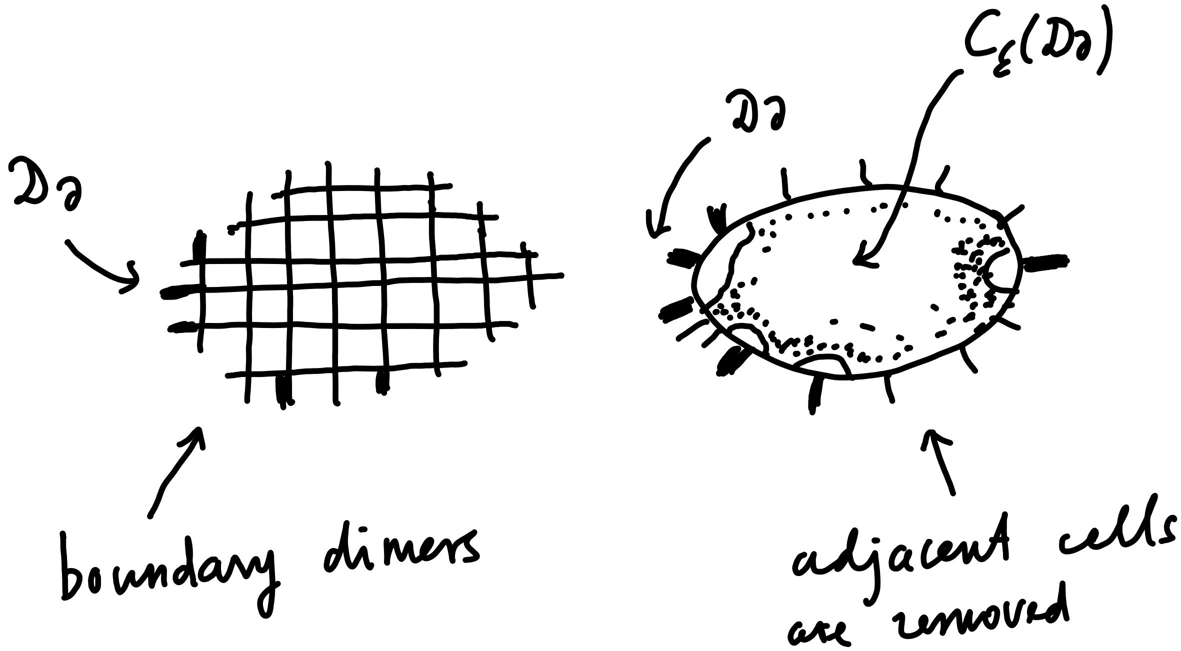
- $\mathcal{L}$ = lattice (double periodic graph)



- $\varphi_\varepsilon : \mathcal{L} \hookrightarrow \mathbb{R}^2$
- $D \subset \mathbb{R}^2$ domain connected, simply connected
- $C_\varepsilon = \varphi_\varepsilon(\mathcal{L}) \cap D$

$C_\varepsilon \to \infty$ means $\{\varepsilon_n\}, \varepsilon_n \xrightarrow[n\to\infty]{} 0$

Boundary conditions



Dimer exclusion rule:

$$\begin{pmatrix}\text{Dimers on outer}\\ \text{edges}\end{pmatrix} \simeq \begin{pmatrix}\text{adjacent } 0,1,2\\ \text{cells are removed}\end{pmatrix}$$

$$Z_{C_\varepsilon}(D_\partial) = Z_{C_\varepsilon(D_\partial)} = \sum_{D \subset C_\varepsilon(D_\partial)} \prod_{e \in D} w(e)$$

$$\mathrm{Prob}(D \subset C_\varepsilon(D_\partial)) = \frac{\prod_{e \in D} w(e)}{Z_{C_\varepsilon}(D_\partial)}$$

$$\mathrm{Prob}(h \in H(C_\varepsilon(D_\partial))) = \frac{\prod_f q_f^{h_D(f)}}{\sum_{h \in H(C_\varepsilon(D_\partial))} \prod_f q_f^{h(f)}}$$

$$\mathrm{Prob}(D) = \mathrm{Prob}(h_D)$$

$\varepsilon_n \to 0$, • stabilizing sequence of boundary conditions

• periodic or scaling weights

$\mathcal{H}(C_\varepsilon(D_\partial))$ = the space of height functions on $C_\varepsilon(D_\partial)$

$h\big|_{\partial C_\varepsilon(D_c)}$ = fixed by $(C_\varepsilon, D_\partial)$

Normalized height function on C_ε

$(n,m) \in \mathcal{L}$ $\qquad\qquad \varphi_\varepsilon(n,m) = (\varepsilon n, \varepsilon m)$

↑ coord. of the fundamental domain

$$\varphi_\varepsilon(\varepsilon n, \varepsilon m) = \varepsilon h(n,m)$$

$$|\varphi_\varepsilon(x+\varepsilon, y) - \varphi_\varepsilon(x,y)| < \varepsilon$$

$$|\varphi_\varepsilon(x, y+\varepsilon) - \varphi_\varepsilon(x,y)| < \varepsilon$$

We are interested in the limit $\varepsilon \to 0$ when $\varepsilon h|_{\partial C_\varepsilon(D_\partial)} \to \eta$ on ∂D

<u>Continuous counterpart</u> of $\mathcal{H}(C_\varepsilon)$

the space of <u>height functions on D</u>:

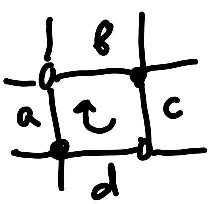
$$H_\eta(D) = \{ \varphi : D \to \mathbb{R} \mid \text{Lipshitz};$$
$$|\varphi(x+t,y) - \varphi(x,y)| \leq t, |\varphi(x,y+t) - \varphi(x,y)| \leq t$$
$$\varphi|_{\partial D} = \eta \}$$

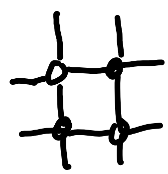
- The thermodynamic limit $\varepsilon_n \to 0$ weights $w(e)$ are double periodic, do not change as $\varepsilon_n \to 0$.

- The thermodynamic limit and scaling, when $\{w(e)\}$ or $\{q_f\}$ change as $\varepsilon_n \to 0$

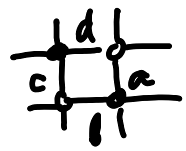
<u>Two examples</u>:

1) square grid, minimal fundamental domain



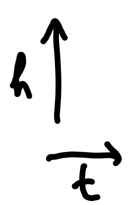


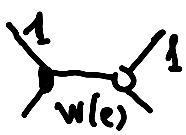
$\to a b^{-1} c d^{-1} = q$



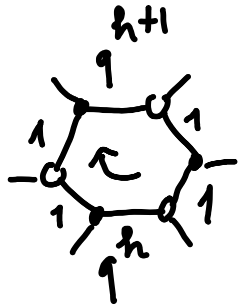
$\to a^{-1} b c^{-1} d = q^{-1}$

2) hexagonal lattice, minimal fund. dom.



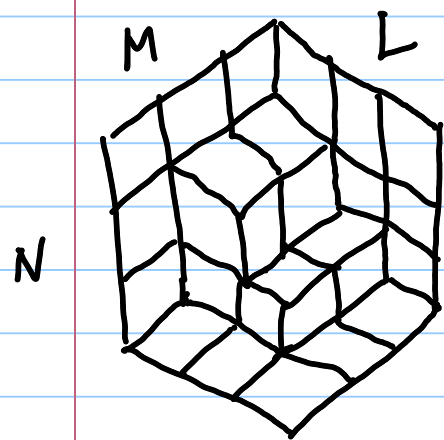


$w(e) = q^h$



$\to q^{-h} q^{h+1} =$

$= q$



π

$$\mathrm{Prob}(\pi) \propto q^{|\pi|}$$

Boundary conditions for $h(x,t)$ =

= the shape of the lattice domain near the boundary

Thermodynamic limit:

$$N : M : L = \underset{\text{fixed}}{A : B : C}, \quad N \to \infty \quad (\varepsilon \to 0)$$

The limit shape phenomenon in dimer models

1) Torus

$$Z^{torus}_{NM}(H,V) = \sum_{D} \prod_{e\in D} w(e)\, e^{H\Delta h(a)}\, e^{V\Delta h(b)} =$$

$$= \frac{1}{2}\left(\pm Pf(A_{K_1}) \pm Pf(A_{K_2}) \pm Pf(A_{K_3}) \pm Pf(A_{K_4})\right)$$

$N\to\infty$, $N:M$ fixed

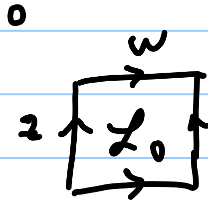
Matrices A_{K_i}: diagonalization by Fourier transform.

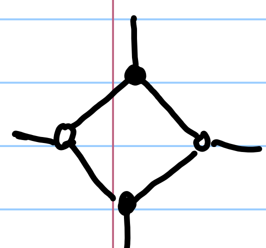
$$Z^{torus}_{NM}(H,V) = \exp\left(-NM\, f(H,V)(1+o(1))\right)$$

$$f(H,V) = \frac{1}{(2\pi i)^2} \oint\limits_{|z|=e^H} \oint\limits_{|w|=e^V} \ln|P(z,w)| \frac{dz}{z} \frac{dw}{w}$$

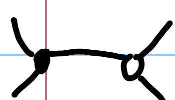
$P(z,w) =$ "spectral polynomial" determined by the fundamental domain of $\mathcal{L}$

$$P(z,w) = \det\left(A_K\big|_{\mathcal{L}_0}(z,w)\right)$$





$$P(z,w) = z + z^{-1} + w + w^{-1} + 1$$



$$P(z,w) = 1 + z + w$$

$$Z^{torus}_{NM}[n,m] = \sum_{h \in \mathcal{H}_{torus}(n,m)} \prod_f q_f^{h(f)}$$

On a torus h is not a function: it may have monodromies along noncontractible cycles.

$$\mathcal{H}_{torus}[n,m] = \{ h \mid \Delta_a h = n , \Delta_b h = m \}$$

Two partition functions are related

$$Z^{torus}_{NM}(H,V) = \sum_{n,m \geq 0} Z^{torus}_{NM}[n,m] \, e^{nH+mV} \qquad (*)$$

as $N \to \infty$, $N : M$, $s = \frac{n}{N}$, $t = \frac{m}{M}$

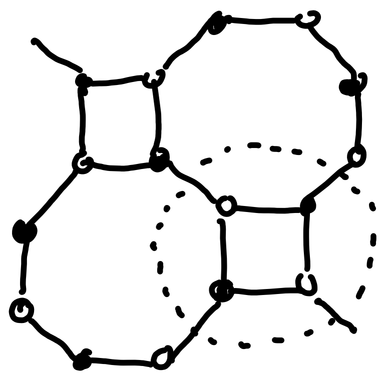
$$Z_{N,M}^{\text{torus}}[n,m] = \exp\left(NM\,\sigma(s,t)(1+o(1))\right)$$

$(*)$ implies

$$f(H,V) = \max_{s,t}\left(Hs + Ht - \sigma(s,t)\right)$$

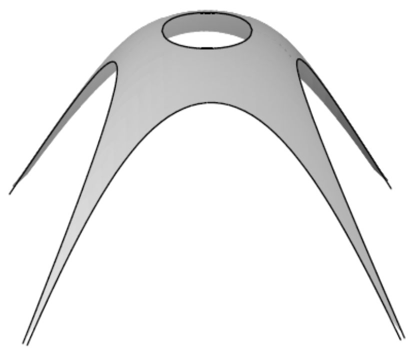
inverse Legendre transform:

$$\sigma(s,t) = \max_{H,V}\left(Hs + tV - f(H,V)\right)$$

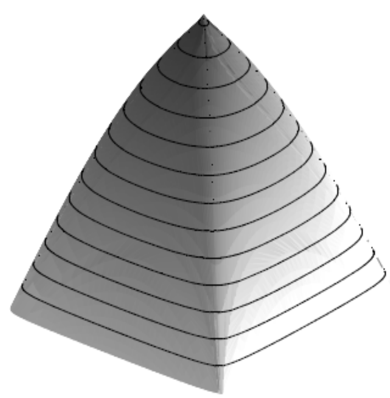


$$P(z,w) = z + z^{-1} + w + w^{-1} + 5$$

Kenyon
Okounkov
Sheffield

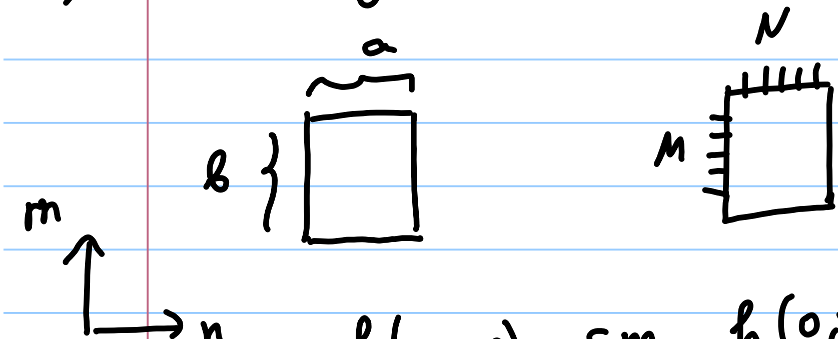


f



σ

2) Rectangular domain



$$h(m,0) = sm, \quad h(0,n) = tn$$

$$h(m,N) = tN + sm, \quad h(M,n) = sM + tn$$

$N = \frac{b}{\varepsilon}$, $M = \frac{a}{\varepsilon}$, with s, t fixed

$\varepsilon \to 0$

Lemma (Cohn, Kenyon, Propp) as $\varepsilon \to 0$

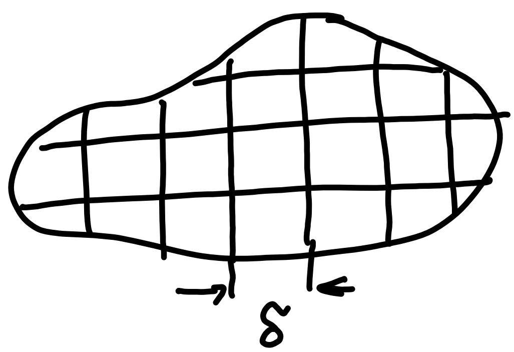
$$Z_{N,M}(n,m) = \exp\left(\frac{ab}{\varepsilon^2}\,\sigma(s,t)(1+o(1))\right)$$

where

$$\sigma(s,t) = \lim_{\varepsilon \to 0} \varepsilon^2 \ln\left(Z^{\text{torus}}_{N,M}(n,m)\right)$$

3) <u>The limit shape phenomenon</u>

Heuristic arguments:



Partition D into small regions of linear size δ

$$Z_{C_\varepsilon(D_\partial)} = \sum_{\substack{\text{over}\\ \text{boundary height}\\ \text{functions (on cuts)}}} \prod_{\substack{\text{over}\\ \text{small regions}}} Z_{C_\varepsilon^{(\alpha)}}(h_\partial)$$

$$\simeq \sum_{\substack{\text{over}\\ \text{boundary height}\\ \text{functions (on cuts)}}} \exp\left(\frac{1}{\varepsilon^2} \sum_{\substack{\text{over}\\ \text{small regions}}} \Delta a \Delta b \, \sigma\left(\frac{\Delta_a h}{\Delta a}, \frac{\Delta_b h}{\Delta b}\right)\right)$$

$$\simeq \int_{\varphi|_{\partial D} = \varphi_\partial} \exp\left(\frac{1}{\varepsilon^2} \iint_D \sigma(\partial_x \varphi, \partial_y \varphi)\, dx\, dy \right)$$

$$\simeq \exp\left(\frac{1}{\varepsilon^2} \iint_D \sigma(\partial_x \varphi_0, \partial_y \varphi_0)\, dx\, dy \right)$$

φ_0 = the minimizer of

$$\boxed{S[\varphi] = \iint_D \sigma(\partial_x \varphi, \partial_y \varphi)\, dx\, dy}$$

on the space $\mathcal{H}(D)$ (Lipshiz)

with boundary condition $\varphi|_{\partial D} = \varphi_\partial$

$$\varphi_\partial = \lim_{\varepsilon \to 0} \varepsilon h \Big|_{\partial C_\varepsilon(D_\partial)} ,$$

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln Z_{C_\varepsilon(\partial D)} = \iint_D \sigma(\partial_x \varphi_0, \partial_y \varphi_0)\, dx\, dy$$

<u>Thm</u> (Cohn, Kenyon, Propp) This is true and

$$\varepsilon h(\varepsilon n, \varepsilon m) \longrightarrow \varphi_0(x, y)$$

as $\varepsilon \to 0$ (in probability).

[For hexagonal lattice, uniform distribution]

<u>Proof</u>. More precise version of the above arguments.

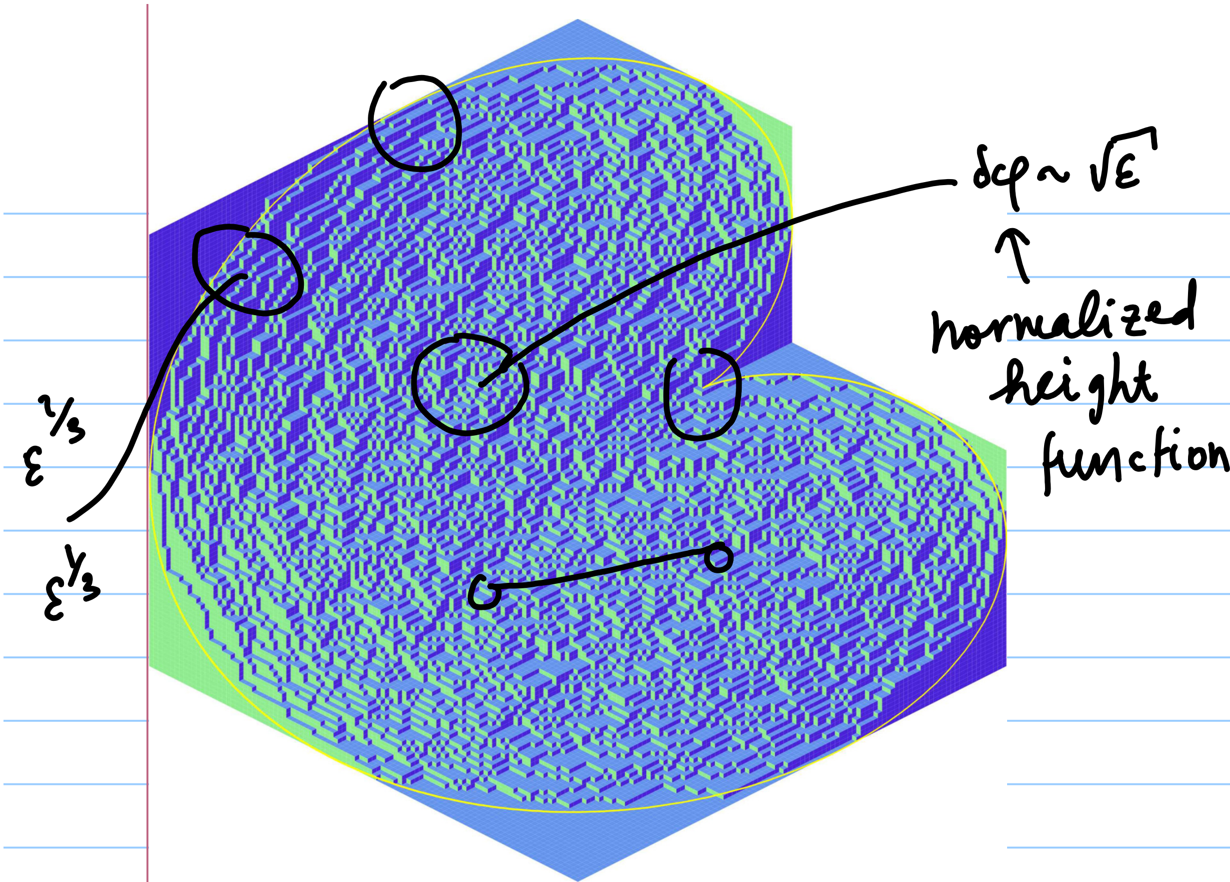
Stonger version of this theorem:

$$\mathbb{E}\Big[\varepsilon h(\varepsilon n_1, \varepsilon m_1) \cdots \varepsilon h(\varepsilon n_k, \varepsilon m_k)\Big] \longrightarrow$$

$$\longrightarrow \varphi_0(x_1, y_1) \cdots \varphi_0(x_k, y_k)$$

For the proofs in various cases see Borodin Ferrari [2008], Petrov [2015] Bufetov Knizel [2016] ....

Fluctuations around limit shapes. Next lecture


$\delta\varphi \sim \sqrt{\varepsilon}$
normalized height function
$\varepsilon^{2/3}$
$\varepsilon^{1/3}$