Limit shapes in integrable models in statistical mechanics

Lect 1. Local Models in Statistical Mechanics
Dimer models

Lect 2. The Thermodynamic Limit in Dimer models and the Limit Shape Phenomenon

Lect 3. Fluctuations around the limit shape and correlation functions in dimer models.

Lect 4. The 6-vertex model. Yang-Baxter equation, commutativity and the spectrum of transfer-matrices

Lect 5. The thermodynamic limit. Torus, cylinder, limit shapes. Variational principle, its Hamiltonian version. Infinitely many conservation laws.

Lect 6. Markov properties of the transfer-matrix and the stochastic point. Limit shapes for the stochastic 6-vertex model. Fluctuations.

Lect 1.

"Lattice" models in statistical mechanics

dim = 2 case

"Space time"

$$C = C_0 \sqcup C_1 \sqcup C_2 \text{ - cell complex}$$

vertices $\quad$ edges $\quad$ faces (2-cells)

States:

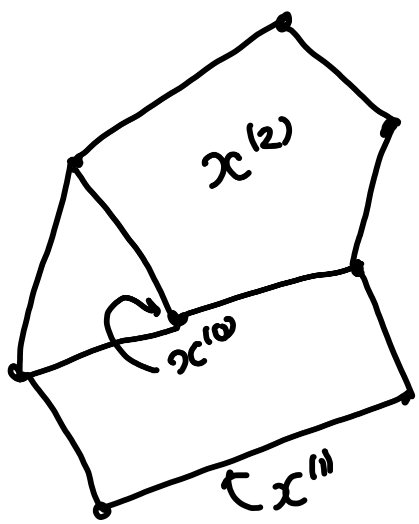
- State bundle

$$\begin{array}{ccccc} X & \xleftarrow{\ \text{discrete}\ } & X_c & \longleftarrow & \left\{\begin{array}{l}\text{space of}\\ \text{states}\\ \text{on } c\end{array}\right. \\ \big\downarrow \pi & & \big\downarrow & & \\ C & & c & & \end{array}$$

- States on C = sections of X:

$$\begin{tikzcd} X \arrow[d, "\pi"] \\ C \arrow[u, bend left, "x"] \end{tikzcd} \qquad x : c \longmapsto x(c)$$

$x(c)$ = state on c



- Boltzmann weights

$$W : \Gamma(X, C) \to \mathbb{R}_{\geq 0}$$

$$x \xmapsto{W} W(x) \geq 0$$

- Physical meaning:

$$W(x) = \exp\left(-\frac{E(x)}{kT}\right)$$

$E(x)$ = the energy of x

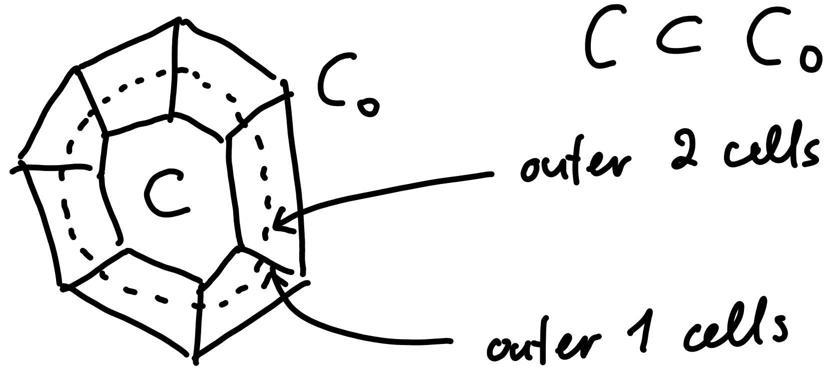
- Probability distribution:

$$\mathrm{Prob}(x) = \frac{1}{Z} W(x)$$

$$Z = \sum_{\substack{\text{all } x \text{ with} \\ \text{given boundary} \\ \text{conditions}}} W(x)$$

Boundary condition

Assume C_0 is planar (or a cell approximation of a surface).

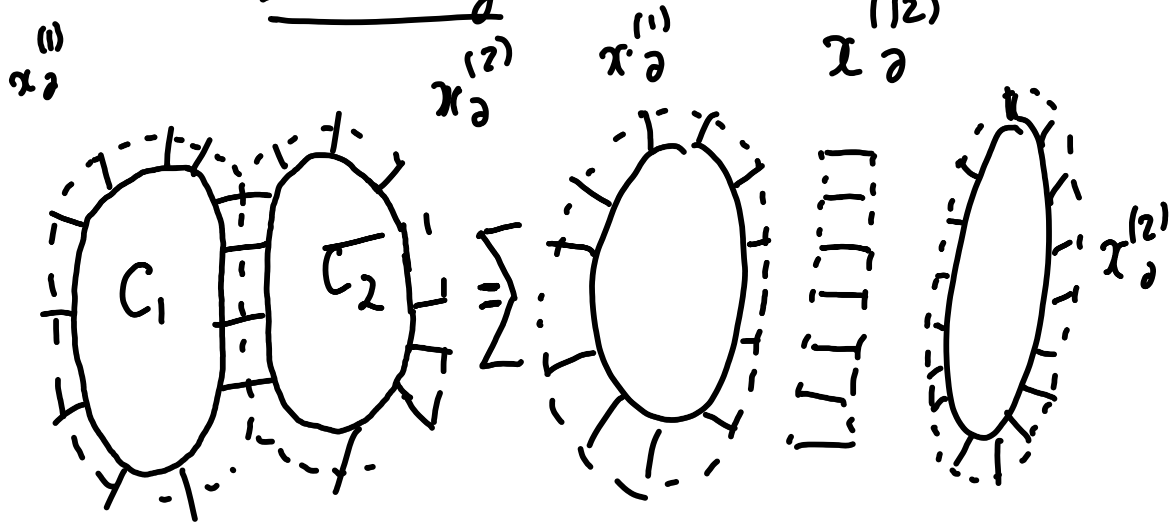


$C \subset C_0$

Boundary states (x_∂) = states on outer 1-cells and 2-cells

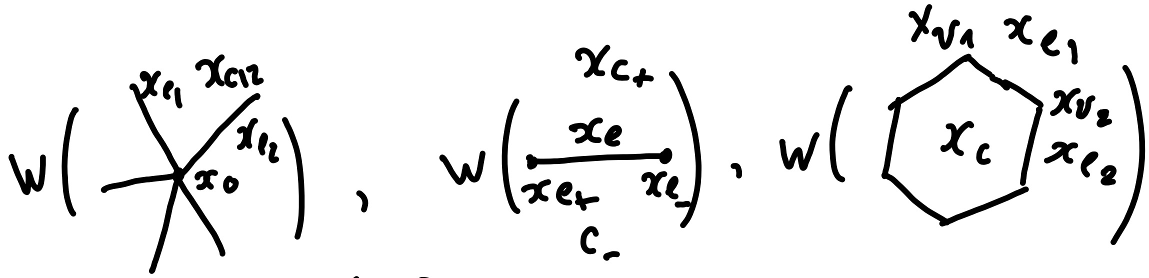
$$\underset{C \subset C_0}{Z}(x_\partial) = \sum_{\substack{x \text{ on } C \\ \text{states on } C}} W(x, x_\partial)$$

Locality



$$Z_{C_1 \cup_I C_2}(x_\partial^{(1)}, x_\partial^{(2)}) = \sum_{x^{(12)}} Z_{C_1}(x_\partial^{(1)}, x_\partial^{(12)}) \cdot w(x_\partial^{(12)})\, Z_{C_2}(x_\partial^{(2)}, x_\partial^{(12)})$$

"Minimal" (local) weights: $\left\{\begin{matrix}\text{depend on the} \\ \text{star of } c\end{matrix}\right\}$

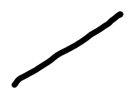


($\mathbb{R}^2$ is oriented)

Examples:

1) Dimer models:

- states on edges, occupied or nonoccupied (by a dimer)

 or 

- $$w(D) = \begin{cases} 0 & \text{if two dimers meet at a vertex} \\ 0 & \text{if } \exists \text{ a vertex not adjacent to a dimer edge} \\ \prod_{e \in D} w(e) & \text{otherwise} \end{cases}$$

$w : E(C) \to \mathbb{R}_{>0}$ given weights

Note; weights can be zero

2) Ising model

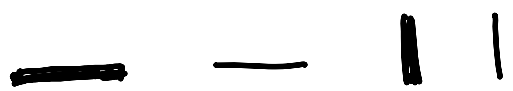
- States on vertices $(+,-)$
- $w(\sigma) = \exp\left(-\sum_{e \in C_1} J_e \sigma_{e_+} \sigma_{e_-}\right)$

Note: all $w(\sigma) > 0$

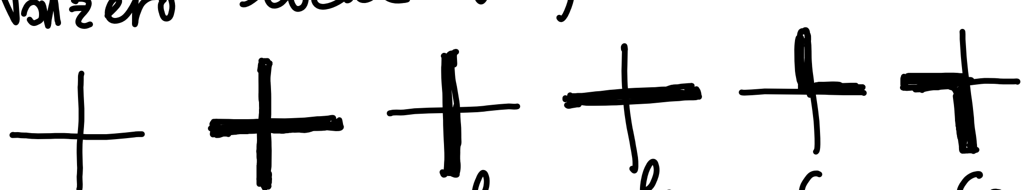
3) The 6-vertex model

C = square grid, oriented

- States on edges: occupied by a path or not occupied by a path



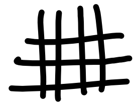
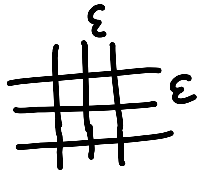
- Nonzero local weights:



a_1 a_2 b_1 b_2 c_1 c_2

Note: other weights are zero

Van Hove theorem

$\mathcal{L} =$  , $\varphi_\varepsilon(\mathcal{L}) \hookrightarrow \mathbb{R}$ 

$C_\varepsilon = \varphi_\varepsilon(\mathcal{L}) \cap \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$ is fixed

$|C_\varepsilon| \to \infty$, means $\varepsilon \to 0$

Assume :

1. $w(x)$ is also <u>periodic</u> and $w(x)$ <u>do not change</u> as $\varepsilon \to 0$
2. <u>$w(x) \neq 0$ for all x</u>
3. <u>X is compact</u>

<u>Thm</u> (VH, see Ruelle)

$$f = -\lim_{|C| \to \infty} \frac{1}{|C|} \ln\left(Z_C(x_\partial) \right)$$

exists and does not depend on boundary conditions (free energy per site)

True for Ising model

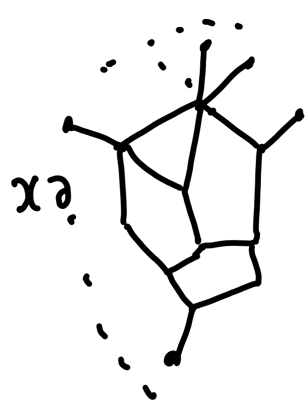
Not true for dimers and 6-vertex: the limit exists but depends on boundary conditions.

<u>Simplest example</u>: Gaussian model

$X = \mathbb{R}^{V(C)}$,

positive definite

$$w(x) = \exp\Big(-\frac{1}{2}\sum_{v,v'} E(v,v')\, x_v x_{v'}\Big)$$



$$Z(x_\partial) = \int_{x_{int}} w(x)\, dx^{|V_{int}(C)|}$$

Gaussian integral

$$Z(x_\partial) = \frac{(2\pi)^{|V_{inner}|}}{\sqrt{\det(E_{Dirichlet})}} \cdot \exp\left(-\frac{1}{2}\sum_{v,v'\in\partial C} x_v x_{v'} (E^{-1})_{vv'}\right)$$

if $\{E(v,v')\}$ = discretized Laplacian on C_ε

$\varepsilon \to 0$

$\{E\} \to \Delta$, $x_\partial \to \eta$

$$Z(x_\partial) = \exp\left(\frac{1}{\varepsilon^2}\int_{\partial D}\int_{\partial D} \eta(x)\eta(y)G(x,y)\,dx\,dy\,(1+o(1))\right)$$

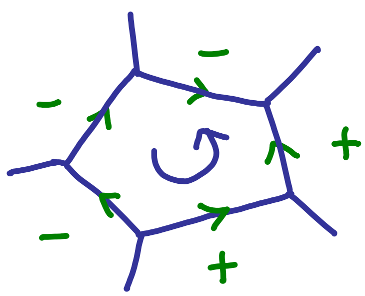
Depends on boundary conditions

(local space of states is noncompact, $\mathbb{R}$)

Kastelyn solution to planar dimer models

1. K-orientations of $\Gamma \subset \mathbb{R}^2$ (spin struct.)

(Kuperberg; Cimasoni, N.R.)

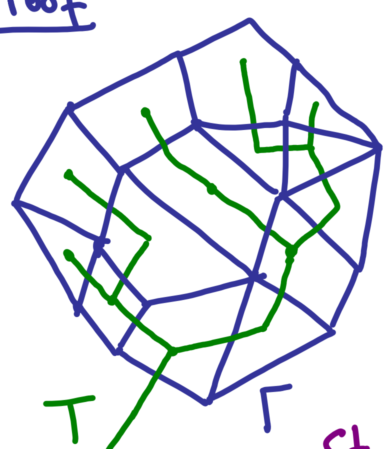


$$\prod_{e \in \partial f} \varepsilon_f(e) = -1$$

for each 2-cell

Prop. Such orientations exist.

Proof



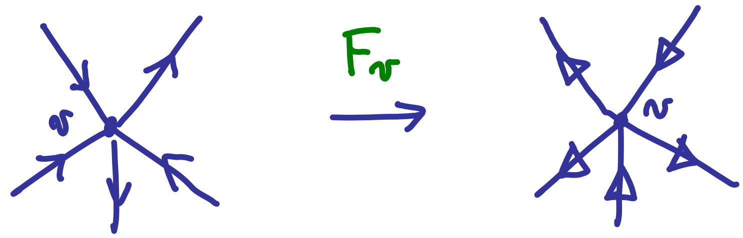
$\Gamma \subset \mathbb{R}^2$ graph, $T =$

= a spanning tree on $\Gamma^\vee$

Step 1 Orient all edges non intersecting T arbitrary

Step 2 Going from leaves to the root orient remaining edges s.t. each face is K-oriented. ∎

$K \sim K'$ if K' obtained from K by a sequence F_v:



Theorem. For each $\Gamma \subset \mathbb{R}^2$ $\exists!$ equivalence class of K-orientations.

2. **K-matrix** Given $\{w(e) \geq 0\}_{e \in E(\Gamma)}$

$$A^K : \mathbb{R}^{V(\Gamma)} \longrightarrow \mathbb{R}^{V(\Gamma)}$$

(Dirac operator)

$$A^K_{vv'} = w(v,v')\, \varepsilon^K_{vv'} \,,$$

$$\varepsilon^K_{vv'} = \begin{cases} 1, & v \longrightarrow v' \\ -1, & v \longleftarrow v' \\ 0, & v \not\longrightarrow v' \end{cases}$$

3. <u>Theorem</u> (Kastelyn 1961, Fisher 1961)

$$Z_\Gamma = \sum_{D \subset \Gamma} \prod_{e \in D} w(e) = \left| Pf\left(A^K\right) \right| \qquad \text{(Kast)}$$

Recall: $A_{ij} = -A_{ji}$, $i,j = 1, \dots, N$

$$Pf(A) = \frac{1}{2^{\frac{N}{2}} \left(\frac{N}{2}\right)!} \sum_{\sigma \in S_N} (-1)^\sigma A_{\sigma_1\sigma_2} A_{\sigma_3\sigma_4} \cdots A_{\sigma_{N-1}\sigma_N}$$

$$= \sum_{\substack{\text{perfect matchings} \\ \text{on } (1,2,3,\dots,N)}} (-1)^{\sigma_m} \underbrace{A_{i_1j_1}}_{\text{"}w(e_{ij})\,\varepsilon^K_{i,j}} A_{i_2j_2} \cdots A_{i_{\frac{N}{2}} j_{\frac{N}{2}}}$$

$$\sigma_m = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & N-1 & N \\ i_1 & j_1 & i_2 & j_2 & \cdots & i_{\frac{N}{2}} & j_{\frac{N}{2}} \end{pmatrix}, \text{ represent. } m$$

$$m = \{\sigma \in S_N\} / \left(i_a \leftrightarrow j_a ,\ (i_a j_a) \leftrightarrow (i_b j_b)\right)$$

Recall: $\sigma_e(\mathcal{D}) = \begin{cases} 1, & e \in \mathcal{D} \\ 0, & e \notin \mathcal{D} \end{cases}$

$$\langle \sigma_{e_1} \cdots \sigma_{e_k} \rangle = \frac{\sum_{\mathcal{D}} \prod_{e \in \mathcal{D}} w(e)\, \sigma_{e_1}(\mathcal{D}) \cdots \sigma_{e_k}(\mathcal{D})}{Z}$$

$$= w(e_1) \cdots w(e_k) \frac{\partial}{\partial w(e_1)} \cdots \frac{\partial}{\partial w(e_k)} \log Z$$

Corollary of (Kast)

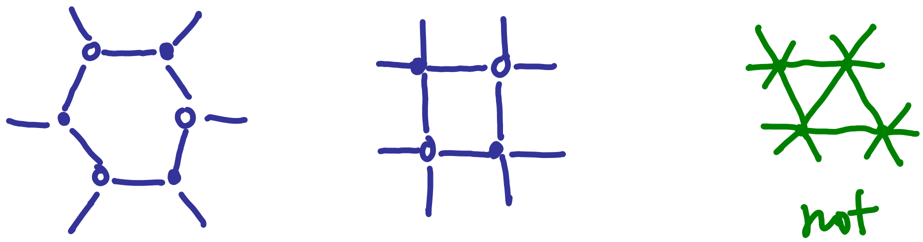
$$\langle \sigma_{i_1 j_1} \sigma_{i_2 j_2} \cdots \sigma_{i_k j_k} \rangle = \mathrm{Pf}\left(\left((A^K)^{-1} \right)_{ab} \right)$$

$$a, b \in \{ i_1, j_1, i_2, j_2 \cdots i_k j_k \}$$

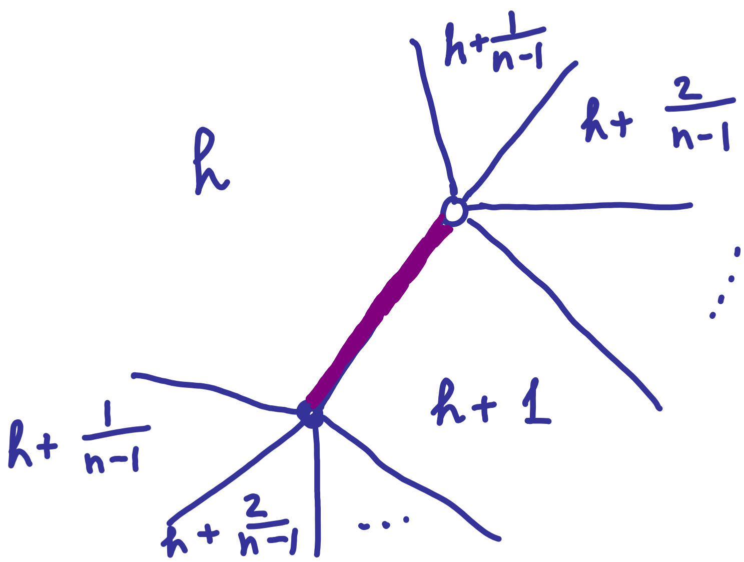
Combinatorics $\rightarrow$ Linear algebra !

Dimers and random surfaces

$\Gamma \subset \mathbb{R}^2$, 2-cell complex, bipartite

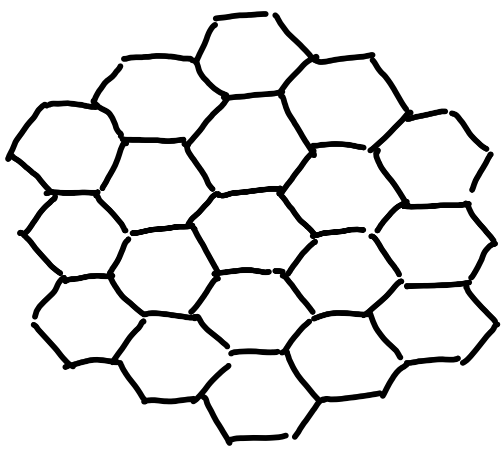


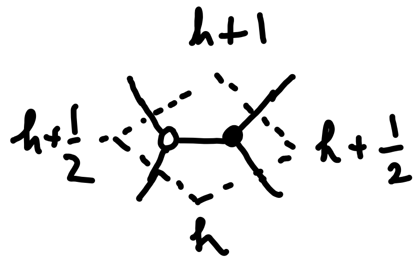
The height function $\quad h_D(f)$

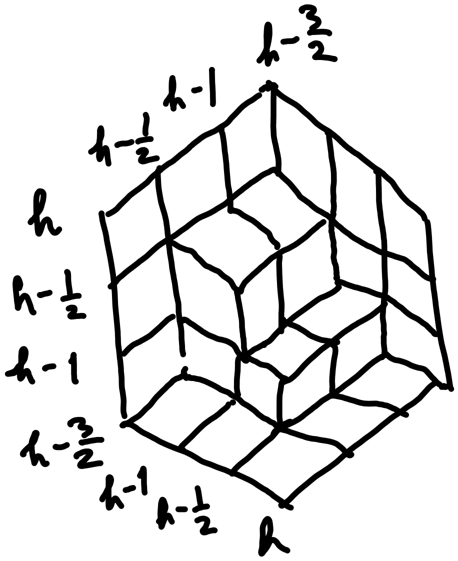


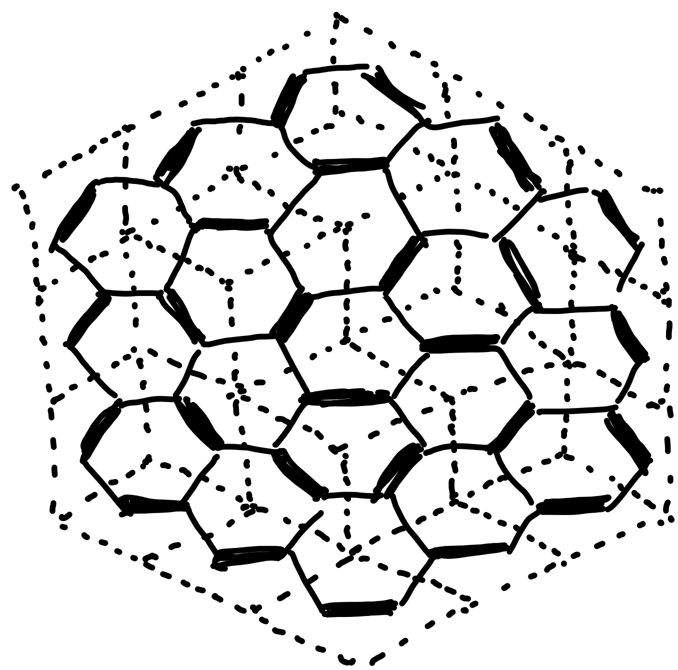
height function = discrete surface over Γ

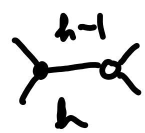
The space of height functions
(hexagonal lattice)







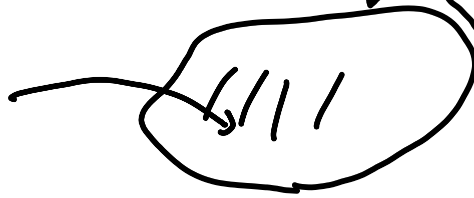




(1,1,1) projection

$$-1+\frac{1}{2}-1+\frac{1}{2}-1+\frac{1}{2}=-3/2$$

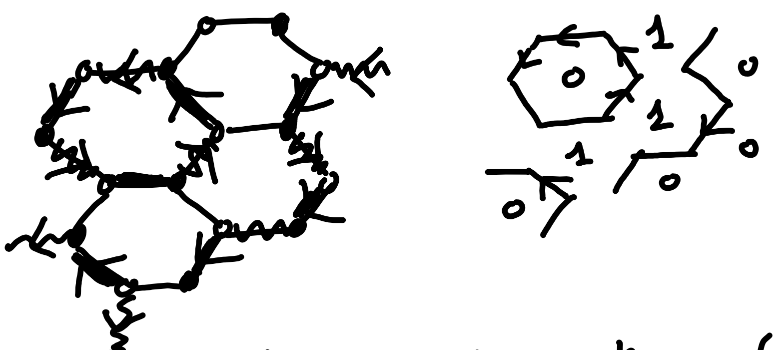
Thm. $h_D\big|_{\partial C}$ does not depend on D.

h_D  boundary value is determined by C

The space of height functions on C

$$H_C = \Big\{ \text{fncns on faces of } C \;\Big|\; h \diagdown h+1 \;\text{ or }\; h \diagdown h+\tfrac{1}{n} \Big\}$$

Remark. Relative height function



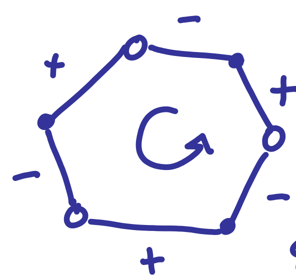
- $h_{D_1,D_2}(f) = \dfrac{n}{n+1}\left(h_{D_1}(f) - h_{D_2}(f)\right)$

- $h_{D_1,D_2}(f)\big|_{\partial C} = 0$

Prob. measure on dimers $\longleftrightarrow$ Prob. measure on height fncns

$$\left\{\begin{array}{l}\text{Edge weights} \\ w(e) \\ e \in E(\Gamma)\end{array}\right\} \Big/ \left\{\begin{array}{c}\text{gauge} \\ w(e) \mapsto S(e_+)\, w(e)\, S(e_-)\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{face weights} \\ q_f \\ f \in F(\Gamma')\end{array}\right\}$$

$$q_f = \prod_{e \in \partial f} w(e)^{\varepsilon(e,f)},$$



$\varepsilon(e,f) =$ = relative orientation

q_f = "gauge invariant"

Lemma

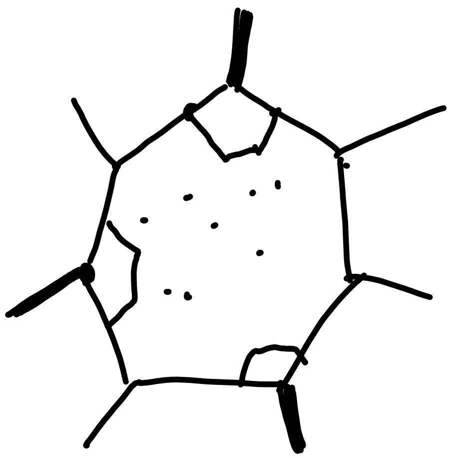
$$\frac{\prod_{e \in D} w(e)}{\sum_{D'} \prod_{e \in D'} w(e)} = \frac{\prod_f q_f^{h_D(f)\frac{n+1}{n}}}{\sum_h \prod_f q_f^{h_D(f)\frac{n+1}{n}}},$$

Proof. $\displaystyle\prod_{e\in D_0} w(e)^{-1} \prod_{e\in D} w(e) = \prod_{C} \prod_{e\in C} w(e)^{\varepsilon(e,C)}$

$$= \prod_f q_f^{\,h_{DD_0}(f)} = \prod_f q_f^{-\frac{n+1}{n} h_{D_0}(f)} \prod_f q_f^{\frac{n+1}{n} h_D(f)}$$

$\blacksquare$

Boundary conditions



dimers on outer edges = dents around corresponding boundary vertices of C

$$\mathrm{Prob}(D, D_\partial) = \frac{\prod_{e\in D} w(e)}{\sum_{D \text{ on } C \setminus \{D_\partial\text{-dents}\}} \prod_e w(e)},$$

$$\mathrm{Prob}(h;\partial h) = \frac{\prod_{f\in C\setminus\{\text{dents}\}} q_f^{h(f)}}{\sum_{\substack{\text{height} \\ \text{functions on } C\setminus\{\text{dents}\}}} \prod_f q_f^{h(f)}}$$

bijectively correspond to ∂h

<u>Exact solution (integrability):</u>

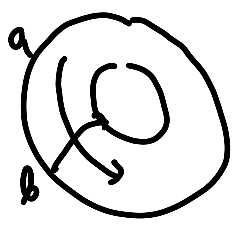
- the partition function (the denominator)
- correlation functions

are given by Kasteleyn Pfaffians.

Remark. Surface graphs $C = (\Gamma \subset \Sigma_g)$.

Assume Γ is bipartite.

Local rules for h_D define a function only on Σ with branch cuts along cycles generating $H_1(\Sigma)$. Extra weights



$$z_c = \exp(H_c)$$

$$Z_C = \sum_{D \subset \Gamma} \prod_{e \in D} w(e) \prod_{c \subset \Gamma} z_c^{\Delta h_D(c)}$$

$\Delta h_D(c)$ = the change of h_D along c

Thm (Kasteleyn 1963, Tesler, 2000, Galuccio, Loebl 1999, Cimasoni, R. 2006)

$$Z_C = \frac{1}{2^g} \sum_{[K]} \mathrm{Arf}(q^K_{D_0}) \, \mathrm{Pf}(A^K)$$

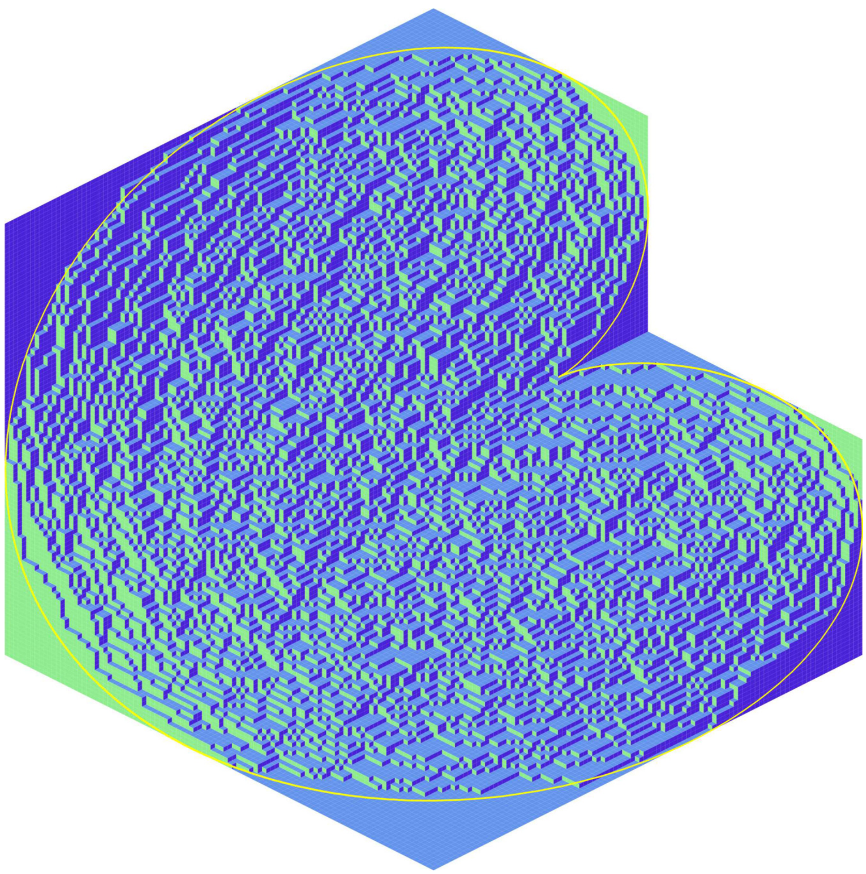
A^K = Kasteleyn matrix (twisted by $\{z_c\}$)
K = Kasteleyn orientation, $[K]$ its eq.cl.
D_0 = reference dimer configuration
$[K, D_0]$ defines spin structure on Σ_g
(Kuperberg)
$q^K_{D_0}$ = the quadratic form corresponding to this spin structure

Corollary

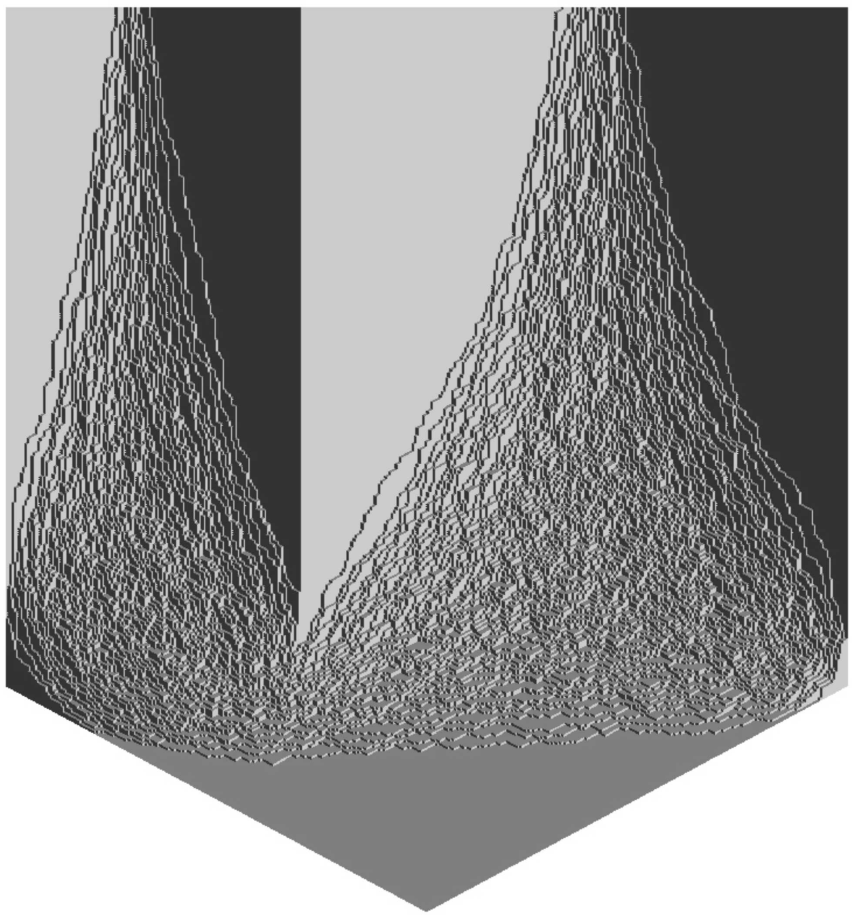
$$Z^{torus} = \frac{1}{2}\left(\pm Pf(A_1) \pm Pf(A_2) \pm Pf(A_3) \pm Pf(A_4)\right)$$

Next two lectures:

$|C_\varepsilon| \to \infty$ limit



uniform, $w(e) = 1$



$q^{|\pi|}, \quad q = e^{-\varepsilon}, \quad \varepsilon > 0$