Lecture 6: Multivariate generating functions

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Minerva Lectures at Columbia University

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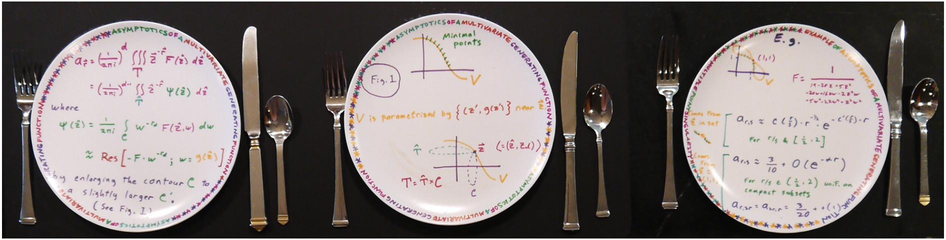
Dinner at the Pemutzle house



Rational series: examples and phenomena

One variable

Let

$$f(z) = \frac{p(z)}{q(z)} = \sum_{r \geq 0} a_r z^r$$

be a rational function.

Then f has a partial fraction expansion as a sum of terms of the form $c(1 - tz)^d$ and therefore there is an exact expression for the coefficient a_r, namely

$$a_t = \sum_{(t,d,c)} c \binom{n+d}{d} t^r$$

summed over triples (t, d, c) in the partial fraction expansion. In short, the theory is trivial.

Several variables

Now turn all the indices into multi-indices, which we denote by upper case letters:

$$F(Z) = \frac{P(Z)}{Q(Z)} = \sum_R a_R Z^R .$$

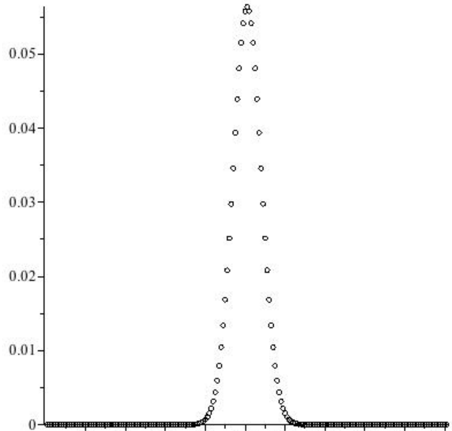
Here, and throughout, this stands for

$$F(z_1, \ldots, z_d) = \frac{P(z_1, \ldots, z_d)}{Q(z_1, \ldots, z_d)} = \sum_{r_1, \ldots, r_d} a_{r_1, \ldots, r_d} z_1^{r_1} \cdots z_d^{r_d} .$$

What phenomena are possible for the behavior of the multi-dimensional array $\{a_R\}$?

Binomial coefficients

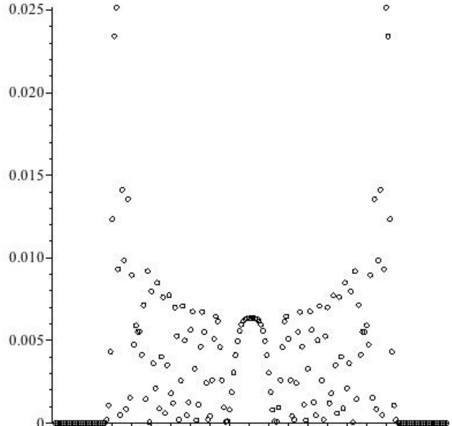
$$F(x, y) = \frac{1}{1 - xy/2 - y/2}$$



shown: $a_{i,200}$

2-D quantum walk

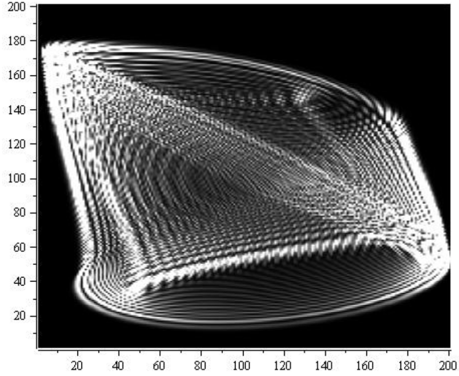
$$F(x,y) = \frac{1}{1-(1-x)y/\sqrt{2}-xy^2}$$



shown: $a_{i,200}$

3-D quantum walk

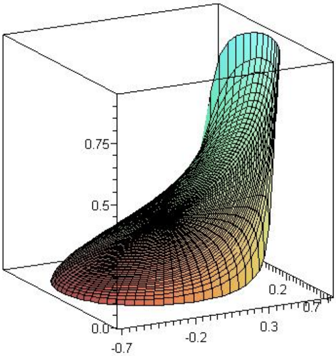
$F(x, y) = \det(I - yMU)^{-1}$, where M is a diagonal matrix of monomials and U is a real orthogonal matrix.



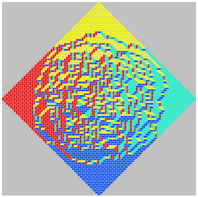
shown: grey scale plot of $a_{i,j,200}$

Dimer tiling

$$F(x,y) = \frac{z/2}{(1-yz)\left[1-(x+x^{-1}+y+y^{-1})z+z^2\right]}$$

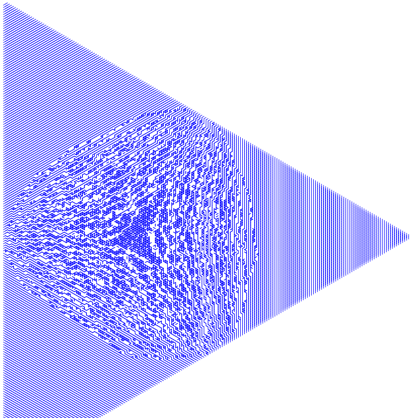


shown: plot of
$(r,s) \mapsto \lim_{t\to\infty} a_{rt,st,t}$

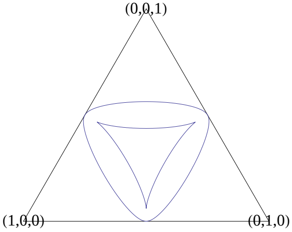


Double-dimer tiling

$F = P/Q$ where $Q(x,y,z) = 63x^2y^2z^2 - 62(x+y+z)xyz - (x^2y^2 + y^2z^2 + z^2x^2) + 62(xy+yz+zx) + (x^2+y^2+z^2) - 63$.



shown: sample tiling, limiting boundaries



Possible behavior along a ray

Let $R = |R| \cdot \hat{R}$ decompose R into magnitude and direction.

The possible behaviors[1] for a_R with $\hat{R}$ roughly fixed are asymptotically, for some rational β, and nonnegative integer γ,

$$a_R = C(\hat{R})|R|^\beta |\log(R)|^\gamma Z(\hat{R})^{-R} \,.$$

Also possible: a finite sum of such terms.

Such formulae hold piecewise, with β and γ constant on each piece and C and Z varying analytically within each piece.

Phase boundaries are algebraic curves;
One expects Airy-type behavior at the boundaries.

[1] Precise statement not proved.

Analytic framework

Cauchy integral

Recall the one variable Cauchy integral

$$a_r = \frac{1}{2\pi i} \int z^{-r} f(z) \frac{dz}{z}$$

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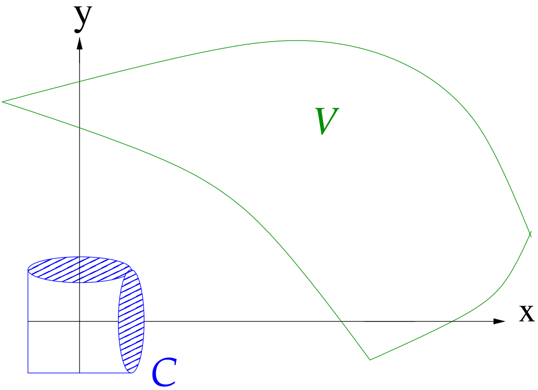
- dZ is the holomorphic volume form;
- integrand is holomorphic in $\mathcal{M} := \mathbb{C}^d \setminus \{Q \prod_{j=1}^d z_j = 0\}$;
- $\mathcal{C}$ is a chain of integration topologically equivalent to the torus $\prod_{j=1}^d \gamma_j$ where γ_j is a circle about the origin in the j^{th} coordinate and the equivalence is in $H_d(\mathcal{M})$.



$\mathcal{M}$ is everything other than V and the coordinate axes

Two levels of accuracy

We want an asymptotic formula for $\mathbf{a_r}$ as $\mathbf{r} \to \infty$ with $\mathbf{r}/|\mathbf{r}| \to \hat{\mathbf{r}}$.

Two levels of accuracy, to be done in two steps:

1. Exponential rate
2. Asymptotic formula

To see how to execute these two steps, recall the univariate case.

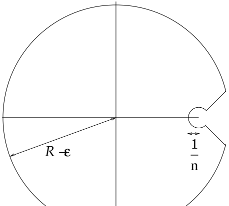
Recall the univariate case

Step 1: let ρ be the radius of convergence.

$$\limsup \frac{\log \mathbf{a_n}}{\mathbf{n}} \leq -\log \rho \, .$$

This completes step one.

Step 2: Use **singularity analysis**. Find the particular singularity $\mathbf{z}_*$ on the radius of convergence and integrate on a contour designed to capture behavior near $\mathbf{z}_*$.



Dominant singularity

Logarithmic coordinates

Instead of a radius of convergence there is a different multi-radius in every direction.

The domain of convergence of a power series or Laurent series is a union of tori

$$\mathbf{T_X} := \{|\mathbf{z}_1| = \mathbf{e}^{\mathbf{x}_1}, \ldots, |\mathbf{z_d}| = \mathbf{e}^{\mathbf{x_d}}\}.$$

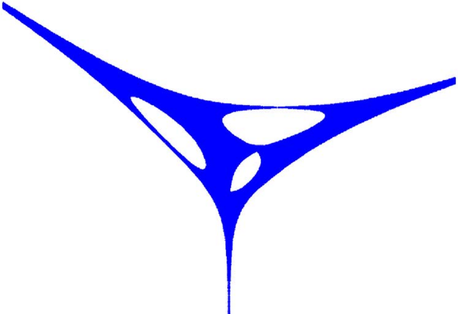
The set of $\mathbf{X} \in \mathbb{R}^{\mathbf{d}}$ for which a series converges is convex.

Map $\mathbb{C}^{\mathbf{d}}$ to $\mathbb{R}^{\mathbf{d}}$ via the log-modulus map $(\mathbf{z}_1, \ldots, \mathbf{z_d}) \mapsto (\log|\mathbf{z}_1|, \ldots, \log|\mathbf{z_d}|)$.

Amoebas

The amoeba of $\mathbf{Q}$ is the set $\{\log|\mathbf{Z}| : \mathbf{Q}(\mathbf{Z}) = 0\}$ where the log modulus map is taken coordinatewise.

On each component of the complement of the amoeba there is a convergent Laurent expansion $\mathbf{P}/\mathbf{Q} = \sum_{\mathbf{R}} \mathbf{a}_{\mathbf{R}} \mathbf{Z}^{\mathbf{R}}$.



Components of the amoeba complement (white regions) are convex.

Imaginary fiber through a point on the boundary

Letting $\mathbf{Z} = \exp(\mathbf{X} + \mathbf{i}\mathbf{Y})$ and sending $\mathbf{X}$ through a component of the complement, to a point on the boundary of the amoeba, the Cauchy integral becomes

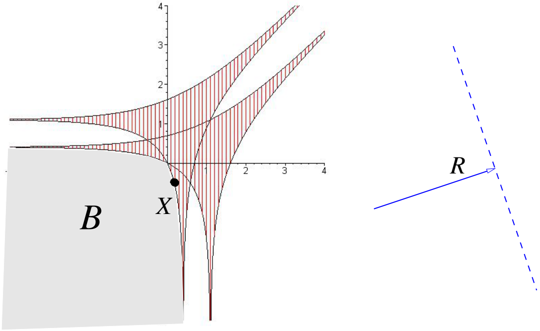
$$\begin{aligned} \mathbf{a}_{\mathbf{R}} &= (2\pi\mathbf{i})^{-\mathbf{d}} \int \mathbf{Z}^{-\mathbf{R}} \mathbf{F}(\mathbf{Z})\, \mathbf{dZ} \\ &= (2\pi)^{-\mathbf{d}} \mathbf{e}^{-\mathbf{R}\cdot\mathbf{X}} \int \exp(-\mathbf{i}\mathbf{R}\cdot\mathbf{Y})\, \mathbf{f}(\mathbf{Y})\, \mathbf{dY} \end{aligned}$$

where $\mathbf{f}(\mathbf{Y}) = \mathbf{F}(\exp(\mathbf{X} + \mathbf{i}\mathbf{Y}))$.

Legendre transform

Choose $\mathbf{X} = \mathbf{X}_*$ to minimize the magnitude of the integrand.

For fixed $\hat{\mathbf{R}}$, this means to make $-\hat{\mathbf{R}} \cdot \mathbf{X}_*$ as small as possible. This is a convex minimization problem (the Legendre transform).



B is a region of the complement corresponding to an ordinary power series;

$\hat{R}$ is given;

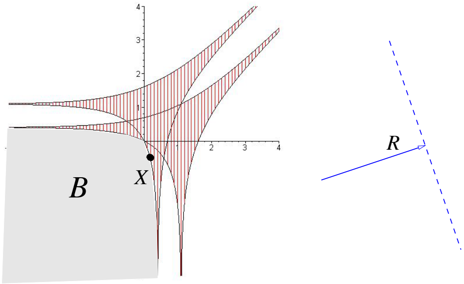
X_* is the minimizing point.

Upper bound

The upper bound is immediate:

$$\limsup \frac{1}{|R|} \log |a_R| \leq -\hat{R} \cdot X_*(R)$$

where $X_*(R)$ is the support point on ∂B normal to R.



Asymptotic evaluation

Contributing points

We turn now to Step 2, namely the asymptotic evaluation of a_R. This is the only way to provide a matching lower bound (in fact the limsup and liminf behavior of the coefficients might not be the same).

Recall that every X in the amoeba of Q is $\log |Z|$ for some at least one $\mathbb{Z} \in \mathcal{V}$. The Cauchy integral near *some* of these is what determines the asymptotics in direction R.

We next discuss the identification of point(s) $\exp(X_* + iY) \in \mathcal{V}$ that are responsible for the coefficient asymptotics in direction R.

Tangent cones

Finally... we get back to hyperbolicity.

Let $T := \mathcal{V} \cap \{\exp(X_* + iY) : Y \in (\mathbb{R}/2\pi\mathbb{Z})^d\}$.

At each $Z \in T$, let $p = p_Z$ be the homogeneous polynomial defined by the leading term of $Q(Z + \cdot)$.

The polynomial p_Z is the **algebraic tangent cone** of Q at Z.

Proposition (Baryshnikov+P 2011)

1. *For every $Z := \exp(X + iY)$ with X on the boundary of the amoeba of Q, the polynomial p_Z is hyperbolic.*
2. *p_Z has a cone of hyperbolicity contaning the support cone B to the amoeba complement at X.*

Family of cones

This is the key construction for evaluating the Cauchy integral.

Theorem (semi-continuous family of cones)

Let p be any hyperbolic homogeneous polynomial and let B be a cone of hyperbolicity for p. There is a family of cones $K(x)$ indexed by the points x at which p vanishes, such that the following hold.

(i) *Each $K(x)$ is a cone of hyperbolicity for the tangent cone p_x.*

(ii) *All of the cones $K(x)$ contain B.*

(iii) *$K(x)$ is* **semi-continuous** *in x, meaning that if $x_n \to x$, then $K(x) \subseteq \liminf K(x_n)$.*

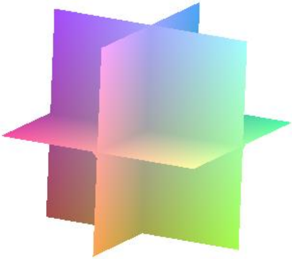
Proof – examples to follow

Step 1: By the previous proposition p_x is hyperbolic. Now show that any vector hyperbolic for p is also hyperbolic for p_x. This result was originally proved by Atiyah-Bott-Gårding (1970); Borcea (personal communication) gave a short, self-contained proof.

Step 2: Pick u in B. Define $K(x)$ to be the cone of hyperbolicity of p_x that contains u. This gives (i) and also (ii) because the construction is the same for any $u \in B$.

Step 3: Prove (iii) by showing that these cones obey a condition similar to that of Whitney stratification. This takes a few geometric steps.

Example: orthant

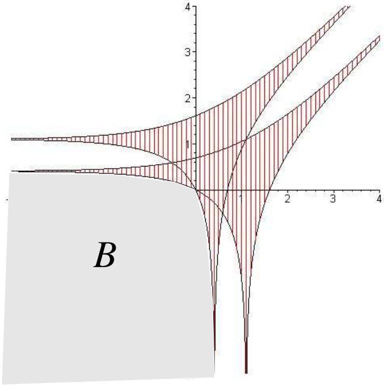


If x is on a 2-D surface then $K(x)$ is the halfspace containing B.

If x is on one of the intersection lines then $K(x)$ is the quarter-space contaning B.

If x is the origin then then $K(x)$ is the octant B.

Example: product of two linear polynomials



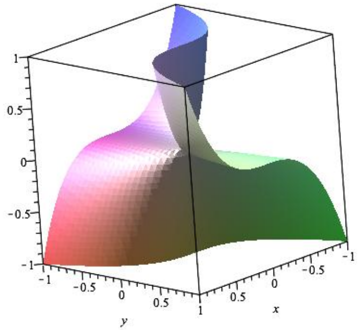
If $x = 1$, the common intersection of the two divisors, then $K(x) = B$.

If x is on only one of the lines then $K(x)$ is a halfspace tangent to one of the two factor amoebas.

Shown: the amoeba for $p = (3z - x - 2y)(3z - 2x - y)$.

The amoeba of the product is the union of the two factor amoebas.

Counterexample: when p is not hyperbolic



Along the line $(0, y, 0)$, $K(y)$ cones of hyperbolicity for p_y can be chosen but are forced to select the positive or negative z direction.

One of these violates semi-continuity for $K(0^+, y, 0)$ while the other choice violates semi-continutiy at $K(0^-, y, 0)$.

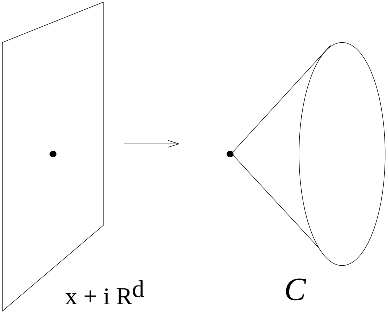
Semi-continuous cones give vector field

Theorem (Morse deformation; BP2011, after ABG1970)

If $\{K(x)\}$ is a semi-continuous family of cones, then a continuous vector field Ψ may be constucted with $\Psi(x) \subseteq K(x)$ for each x.

This allows the chain of integration for the Cauchy integral to be deformed so that the integrand is very small except in a neighborhood of Z.

The deformation is locally projective.



The Cauchy integral and the Riesz kernel

Recall the integral in logarithmic coordinates

$$a_R = (2\pi)^{(1-d)/2} \exp(-R \cdot X) \int \exp -(iR \cdot Y) f(Y)\, dY\, .$$

Pushing the chain of integration from the imaginary fiber outward in a conical manner produces a homogeneous inverse Fourier transform.

Leading asymptotic behavior only depends on leading behavior of the homogenization $1/p_Z$. We recognize the IFT

$$\int_\gamma p_z^{-1} \exp(iR \cdot Y)$$

as the Riesz kernel for the homogeneous hyperbolic polynoimal p_z.

Inverse Fourier transforms

The estimates needed to establish the existence of the integral rely on the projective deformation.

Relating it to previously computed IFT's (in, e.g., [ABG1970]) uses the theory of **boundaries of holomorphic functions**, laid out by Hörmander (1990).

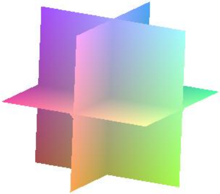
For example, the IFT of a linear function $ax + by + cz$ is a delta function on the ray $\lambda\langle a, b, c\rangle$ in the dual space. (Here the index space $\mathbb{Z}^d$ is the dual space and the real/complex space in which the generating function variables live is the primal space.)

Examples of computed IFT's

Example (orthant)

The IFT of $1/(xyz)$ is the constant 1 on the orthant. This corresponds to the generating function

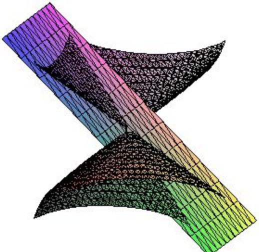
$$\frac{1}{(1-x)(1-y)(1-z)} .$$



Quadratic times linear

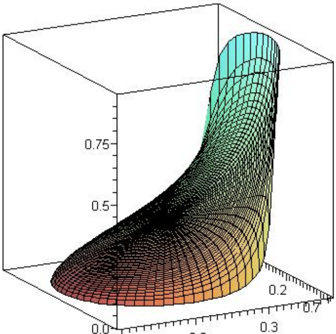
"Quadratic times linear" describes the homogeneous part of the Aztec Diamond probability generating function

$$\frac{z/2}{(1 - yz)\left[1 - (x + x^{-1} + y + y^{-1})z + z^2\right]} .$$

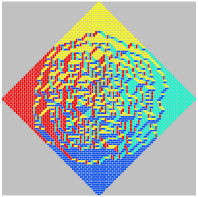


Example: Aztec diamond tilings

IFT is a convolution of a delta function on the ray $(0, \lambda, \lambda)$ with the IFT of a circular quadratic. The quadratic is self-dual, with IFT equal to $t^2 - r^2 - s^2$.



shown: plot of $(r, s) \mapsto \lim_{t \to \infty} a_{rt,st,t}$

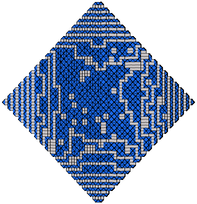


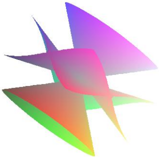
General case

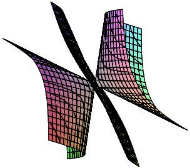
Inverse Fourier transforms for general hyperbolic homogeneous polynomials have not been effectively computed although quite a number have been worked out, e.g., Atiyah-Bott-Gårding (Acta Math. 1970) "Lacunas for hyperbolic differential operators with constant coefficients, I."

More to be done

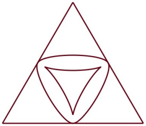
General cubic and quartic integrals are a little trickier to evaluate explicitly than are the quadratics and factored cubics. The so-called fortress tillings are an example of this (work in progress with Y. Baryshnikov).

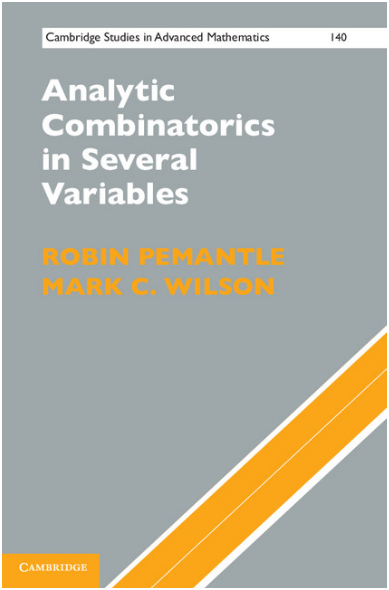






This cubic arises in analysis of the hexahedron recurrence. Its IFT gives a limit shape theorem (work in progress). At present we can describe the feasible region but not the limit statistics within the region. For a particular parameter value, the central collar becomes a plane and the results for Quardatic times Linear apply.

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