Definitions

Definition (stability)

A polynomial q is said to be **stable** if $\mathbf{q}(\mathbf{z}) \neq 0$ whenever each coordinate $\mathbf{z_j}$ is in the strict upper half plane.

Definition (hyperbolicity)

A homogeneous polynomial $\mathbf{p}$ of degree $\mathbf{m}$ is said to be **hyperbolic** in direction $\mathbf{x} \in \mathbb{R}^{\mathbf{d}}$ if $\mathbf{p}(\mathbf{x} + \mathbf{i}\mathbf{y}) \neq 0$ for all $\mathbf{y} \in \mathbb{R}^{\mathbf{d}}$.

Proposition (equivalences)

- *Hyperbolicity in direction* $\mathbf{x}$ *is equivalent to the univariate polynomial* $\mathbf{p}(\mathbf{y} + \mathbf{t}\mathbf{x})$ *having only real roots for all* $\mathbf{y} \in \mathbb{R}^{\mathbf{d}}$.
- *A real homogeneous polynomial is stable if and only if it is hyperbolic in every direction in the positive orthant.*

Cones of hyperbolicity

The real variety $\mathcal{V} := \{\mathbf{p} = 0\} \subseteq \mathbb{R}^{\mathbf{d}}$ plays a special role in hyperbolicity theory.

Proposition (cones of hyperbolicity)

Let ξ be a direction of hperbolicity for $\mathbf{p}$ and let $\mathbf{K}(\mathbf{p}, \xi)$ denote the connectied component of the set $\mathbb{R}^{\mathbf{d}} \setminus \mathcal{V}$ that contains ξ.

- $\mathbf{p}$ *is hyperbolic in direction* $\mathbf{x}$ *for every* $\mathbf{x} \in \mathbf{K}(\mathbf{p}, \xi)$.
- *The set* $\mathbf{K}(\mathbf{p}, \xi)$ *is an open convex cone; we call this a* **cone of hyperbolicity** *for* $\mathbf{p}$.

Stable evolution of PDE's

Let $\mathbf{p}$ be a polynomial in $\mathbf{d}$ variables and denote by $\mathbf{D_p}$ the operator $\mathbf{p}(\partial/\partial\mathbf{x})$ obtained by replacing each $\mathbf{x_i}$ by $\partial/\partial\mathbf{x_i}$.

Let $\mathbf{r}$ be a vector in $\mathbb{R}^{\mathbf{d}}$, let $\mathbf{H_r}$ be the hyperplane orthogonal to $\mathbf{r}$, and consider the equation

$$\mathbf{D_p}(\mathbf{f}) = 0 \tag{1}$$

in the halfspace $\{\mathbf{r}\cdot\mathbf{x} \geq 0\}$ with boundary conditions specified on $\mathbf{H_r}$ (typically, $\mathbf{f}$ and its first $\mathbf{d}-1$ normal derivatives).

We say that (1) evolves **stably** in direction $\mathbf{r}$ if convergence of the boundary conditions to 0 implies convergence[1] of the solution to 0.

[1] Uniform convergence of the function and its derivatives on compact sets.

Gårding's Theorem

Theorem ([Går51, Theorem III])

The equation $D_p f = 0$ evolves stably in direction r if and only if p is hyperbolic in direction r.

Let us see why this should be true.

Inequalities that make this work well

Suppose $\phi \in C^3$ is defined on the interior of a convex cone K, is homogeneous of degree $-m$, and goes to infinity at ∂K.

If $F := \log \phi$ satisfies

$$|D^3 F(x)[h,h,h]| \leq 2\left(D^2 F(x)[x,x]\right)^{3/2}$$

then ϕ serves as a barrier function. If, in addition,

$$|DF(x)[h]|^2 \leq m D^2 F(x)[h,h]|$$

then the interior method will converge well.

van der Waerden conjecture

Theorem (van der Waerden conjecture)

The minimum permanent of an $n \times n$ doubly stochastic matrix is $n!/n^n$, uniquely obtained when all entries are $1/n$.

This was proved by Falikman (value of the minimum) and Egorychev (uniqueness) in 1981.

Gurvits (2008) found a much simpler proof using stability. Let $\mathcal{C}_n$ be the set of homogeneous polynomials of degree n in n variables with nonnegative coefficients.

Permanent as a stable polynomial in $\mathcal{C}_n$

If A is an $n \times n$ matrix then its permanent may be represented as a mixed partial derivative of a homogeneous stable polynoimal.

Define

$$p_A(x) := \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j \, .$$

Proposition (Gurvits 2008)

The polynomial p is stable and

$$\mathrm{per}\,(A) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p_A(0, \ldots, 0) \, .$$

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