# Lecture 1: Zeros of random polynomials and power series 

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## Overview of six lectures

I Zeros and coefficients of random polynomials
II What the location of zeros tells us about coefficients of univariate polynomials
III Multivariate theory:
Boolean variables and the strong Rayleigh property
IV Multivariate Applications:
random trees, determinantal measures and sampling
$\checkmark$ Hyperbolic polynomials
VI Coefficient asymptotics for multivariate generating functions

## I: Zeros and coefficients of random polynomials and series

Mostly throughout these lectures we will be placing hypotheses on the locations of zeros and asking questions about the behavior of the coefficients.

We start, however, with a different approach, asking about the locations of zeros of random polynomials and series.

- Random polynomials with specified coefficient law
- Random polynomials whose zeros have specified law: behavior under differentiation
- Random series whose zeros have specified law: behavior under repeated differentiation


## II: Coefficients of polynomials with restricted zero set

This lecture concerns coefficient properties of polynomials with only real roots (or more generally, roots forbidden to lie in some region).

- Limit theorems and inequalities that follow from having only real roots, or having no roots in some specified region
- Examples from combinatorics and statistical physics
- How to prove real-rootedness: closure properties and Pólya-Schur theory.


## III: Binary variables and strong Rayleigh distributions

This lecture develops the theory of strong Rayleigh distributions for binary valued random variables.

- Background: negative dependence properties of distributions on Boolean lattices
- Stable polynomials
- The multi-affine case: strong Rayleigh distributions
- Closure properties of the strong Rayleigh class
- Stochastic properties of strong Rayleigh measures


## IV: Negatively dependent measures and sampling schemes

The companion lecture to Lecture 3, this lecture is entirely devoted to examples of strong Rayleigh distributions and consequences of the strong Rayleigh property.

- Determinantal measures
- Spanning trees
- Sampling proceudres


## V: Hyperbolic polynomials

The last two lectures concern the geometry of multivariate generating functions and the asymptotic extraction of coefficients.

- Outline of analytic combinatorics in several variables
- Hyperbolicity: definitions and origins
- Theorems and conjectures on determinants
- Amoebas and hyperbolicity lemma
- Convex programming and self-concordant barriers


## VI: Asymptotics for multivariate generating functions

- Amoebas and the exponential order
- Morse theory and contour deformations
- Generalized Fourier transforms
- Examples: random tilings, lattice recursions, quantum walks


# Classical random polynomial models: 

## Random coefficients

## Random coefficients

We begin with a model from a seminal paper by Marc Kac [Kac43].
He considers a polynomial defined by random coefficients and asks how many of the zeros are real. More specifically, what is the expected number of zeros in each real interval?

- Let $a_{0}, a_{1}, \ldots, a_{N}$ be real numbers
- Let $\left\{Z_{j}: 0 \leq j \leq N\right\}$ be IID standard normal
- Let $f$ be the random polynomial $f(x):=\sum_{j=0}^{N} a_{j} Z_{j} x^{j}$

Object: determine the expected number of zeros of $f$ in $[t, t+d t]$.

## Gaussian process

The values of $f$ form a centered Gaussian process, with

$$
K(s, t):=\mathbb{E} f(s) f(t)=\sum_{j=0}^{N} a_{j}^{2}(s t)^{j}
$$

Kac said, roughly (this doesn't require Gaussian assumption):

$$
f \text { has a zero in }[t-\epsilon, t+\epsilon] \Longleftrightarrow|f(t)| \leq \epsilon\left|f^{\prime}(t)\right| .
$$

Sending $\epsilon \rightarrow 0$ and multiplying by $\epsilon^{-1}$, the expectation goes to

$$
\text { (density of } f(t) \text { at } 0) \cdot \mathbb{E}\left[\left|f^{\prime}(t)\right| \mid f(t)=0\right] .
$$

## Gaussian expectation

For any Gaussian pair $(X, Y)$ with covariances $\left[\begin{array}{ll}a & b \\ b & c\end{array}\right]$ the density of $X$ at zero is $1 / \sqrt{a}$ and $\mathbb{E}(|Y| \mid X=0)=\sqrt{\Delta / a}$ where $\Delta$ is the determinant $a c-b^{2}$. Thus

$$
\text { (density of } X \text { at } 0 \text { ) } \cdot \mathbb{E}[|Y| \mid X=0]=\frac{\sqrt{\Delta}}{a}
$$

The vector $\left(f(t), f^{\prime}(t)\right)$ has covariance structure

$$
\left[\begin{array}{cc}
K(t, t) & \left.K_{s}(s, t)\right|_{s=t} \\
\left.K_{s}(s, t)\right|_{s=t} & \left.K_{s t}(s, t)\right|_{s=t}
\end{array}\right] .
$$

Conveniently, $\sqrt{\Delta} / a=\sqrt{\partial_{s t} \log K}$.

## Kac-Rice formula

## Theorem (Kac-Rice formula)

Let $\mu(I)$ denote the expected number of zeros of $f$ in the real interval I. Then $\mu$ is a measure with density

$$
\rho(x)=\frac{1}{\pi} \sqrt{\left.\frac{\partial^{2}}{\partial s \partial t} \log K(s, t)\right|_{s=t=x}} .
$$

## Example: standard Gaussian polynomial

## Example

Let $a_{k}=1$ for all $k$. Plugging in $K(s, t)=\frac{1-(s t)^{N+1}}{1-s t}$ and integrating over $\mathbb{R}$ yields an expected number of real zeros

$$
\int \rho(x) d x=\frac{2}{\pi}(\log N+C+o(1))
$$

for an explicitly evaluatable $C \approx 0.6257 \ldots$

In other words, out of $N$ total zeros, the number of real zeros is only of order $\log N$.

## Random Gaussian series

Now let $\left\{a_{j}: j \geq 0\right\}$ be a sequence in $\ell^{2}$ and define a random infinite series $f(x):=\sum_{j=0}^{\infty} a_{j} Z_{j} x^{j}$.

There are many interesting examples, some of which you can read about in the monograph [HKPV09].

## Example

If $a_{k}=1 / \sqrt{k!}$ then the zeros form a point process invariant under translations in the complex plane.

## Self-intersections of a random curve

A nice recent application of Kac-Rice theory involves the random closed curve which is the image of the complex unit circle under the Gaussian random polynomial $f(x):=\sum_{j=1}^{N} j^{-\beta} Z_{j} z^{j}$.


Figure: Random closed curves with $\beta=2$ (left) and $\beta=1$ (right). $N=100$ and 1000 respectively

## Self-intersections of a random curve

## Theorem ([Riv16])

Suppose the coefficient decay exponent satisfies $\beta>3 / 2$. Then the number of self-intersections of the plane curve is finite and its expectation is $O(\beta-3 / 2)^{-1}$.


The random curve when
$\beta=3 / 2$
(the critical
value) and
$N=1000$.

## Two-parameter Kac-Rice formula

## Proof.

Self-intersections are non-diagonal zeros of the two-parameter process $f(s)-f(t)$. Apply the two-parameter Kac-Rice formula to

$$
f(s, t)=\frac{f\left(e^{i s}\right)-f\left(e^{i t}\right)}{s-t}
$$

## Randomly placed zeros

## Polynomial with zeros IID $\sim \mu$

Let $\mu$ be a probability measure on $\mathbb{C}$. Let $\left\{X_{i}\right\}$ be IID $\sim \mu$ and define the random function $f_{N}$ by

$$
f_{N}(z):=\prod_{j=1}^{N}\left(z-X_{j}\right)
$$

Let $\hat{\mu_{N}}$ denote the empirical measure $N^{-1} \sum_{j=1}^{N} \delta_{X_{j}}$. Of course $\hat{\mu}_{N} \rightarrow \mu$ by the strong law of large numbers.

## Theorem (P+Rivin 2013; Subramanian 2014; Kabluchko 2014)

Let $\nu_{N}$ denote the empirical distribution of the zeros of $f^{\prime}$. Then $\nu_{N} \rightarrow \mu$ as $N \rightarrow \infty$.

## Rouché's Theorem

Three different proofs are known, in increasing generality.
(1) Original proof with Rivin uses Rouché's Theorem to marry most of the zeros of $f$ to nearby zeros of $f^{\prime}$.


## Symmetric functions

The second proof, for measures on the unit circle via symmetric function theory, was in Sneha Subramanian's Ph.D. thesis (2014).
(2) Sneha's argument:

- The coefficients of $f$ are elementary symmetric functions $e_{j}\left(X_{1}, \ldots, X_{N}\right)$.
- The coefficients of $z f^{\prime}$ are $j e_{j}\left(X_{1}, \ldots, X_{N}\right)$.
- For $j=N-o(N)$, the coefficients of $f$ and $f^{\prime}$ are similar; similarity of $e_{j}$ 's implies similarity of power sums; these give the moments, which gives the distribution of the arguments of the zeros, the radii concentrating near 1 .

We will come back to coefficient analysis when discussing random series.

## Potential theory

(3) Potential theoretic argument by Kabluchko [Kab14]:

- As generalized functions, $\frac{1}{2 \pi} \Delta \log |g|=\sum_{z: g(z)=0} \delta_{z}$.
- Integrating with $g=f_{N}^{\prime} / f_{N}$ counts critical points minus zeros.
- $\frac{f_{N}^{\prime}(z)}{f_{N}(z)}=\sum_{j=1}^{N} \frac{1}{z-X_{j}}$.
- Outside of an exceptional set of measure zero,

$$
\log \left|\sum_{j=1}^{N} \frac{1}{z-X_{j}}\right|=o(N) \text { in probability. }
$$

## Poisson random zeros

## The Poisson random series

Let $N:(\Omega, \mathcal{F}, \mathbb{P}) \times(\mathbb{R}, \mathcal{B}) \rightarrow \mathbb{Z}^{+}$be a unit intensity point process.
We use the notation $x \in N$ to denote the event
$\{\omega: N(\omega,\{x\})=1\}$.
For $M>0$, denote by $f_{M}$ the random polynomial defined by

$$
f_{M}(z):=\prod_{x \in N,|x| \leq M}\left(1-\frac{z}{x}\right) .
$$



## The Poisson random series

## Proposition (convergence)

- The limit $\lim _{M \rightarrow \infty} f_{M}$ exists almost surely and defines an entire function $f$ with zeros precisely at points of the Poisson process.
- The law of $f$ is translation invariant up to constant multiple.
- The logarithmic derivative $f^{\prime} / f$ is given by the conditionally convergent symmetric sum

$$
\sum_{x \in N} \frac{1}{z-x}
$$

## Behavior of zeros under differentiation

The zeros of $f^{\prime}$ are a translation-invariant point process. How is this point process related to the original one?

Differentiation smooths zeros and makes them more real.

## Facts:

1. If $g$ is a real polynomial with real zeros, then the minimum distance between consecutive zeros of $g^{\prime}$ is at least the minimum distance betwen consecutive zeros of $g$.
2. If $g$ is a real polynomial with $2 k$ non-real zeros then $g^{\prime}$ has at most $2 k$ non-real zeros.

## Iterated differentiation

What happens to the zeros of $f$ under iterated differentiation?
It was believed that these should even out and approach perfect spacing.


Figure: As time increases ( $y$-direction), the spacings become more even

## Lattice limit

## Theorem (P+Subrmanian 2015)

Let $f^{(n)}$ denote the $n^{\text {th }}$ derivative of $f$, with zero set denoted by $\mathcal{Z}_{n}$. Then $\mathcal{Z}_{n}$ converges in distribution to $U+\mathbb{Z}$, a random translate of the integer lattice by a uniform $[0,1]$ random variable.

In the remaining time, I will sketch the argument.
This invokes a number of the ideas that will come up on the last day, in the discussion of multivariate coefficient extraction.

## Wouldn't it be nice. . .

Think of a nice function whose zeros are a random translate of $\mathbb{Z}$ :

$$
g(z):=\cos (\pi z+U[0,2 \pi]))
$$

Of course there are many others, such as $g(z) e^{\phi(z)}$ where $\phi$ is any entire function.

But wouldn't it be nice if somehow $f^{(n)}$ were converging to $g$ ?
The $z^{r}$ coefficient of $g$ is $\cos \left(U-r \frac{\pi}{2}\right) \frac{\pi^{r}}{r!}$.
Random phase with period 4, and magnitude $\pi^{r} / r$ !.

## ... and indeed it's true!

## Lemma

Let $a_{n, k}:=\left[z^{k}\right] f^{(n)}(z)$. There are random $A_{n}$ and $\theta_{n}$ such that for any fixed $k$, as $n \rightarrow \infty$,

$$
a_{n, k}=A_{n}\left[\cos \left(\theta_{n}-\frac{\pi}{2} k\right)+o(1)\right] \frac{\pi^{k}}{k!} \text { in probability. }
$$

Suprising? Up to some factorials, $a_{n, k}$ is the same as $e_{n+k}\{-1 / x: x \in N\}$, the $(n+k)^{t h}$ elementary symmetric function of the reciprocal roots. The reciprocal roots are themselves a Poisson process with intensity $x^{-2} d x$, on which the functions $e_{m}$ are conditionally convergent sums. So, why are these elementary symmetric functions 4 -period in sign?

## Cauchy formula

Write $a_{n, k}=e_{n+k} \frac{(n+k)!}{n!}$ and use Cauchy's formula to get

$$
a_{n, k}=\frac{(n+k)!}{n!} \frac{1}{2 \pi i} \int z^{-m} f(z) \frac{d z}{z} .
$$

Strategy: Show that the integral comes from contributions near dominant saddle points $s= \pm i(k / \pi)$.

This will give $e_{m} \sim 2 K \Re\left\{s^{-m} f(s)\right\}$ (take $m=n+k$ ). The value $K=\sqrt{\frac{2 \pi}{\phi^{\prime \prime}(s)}}$ is not hard to compute but is not important; the location of the saddles already implies that the amplitude of successive terms decreases by $k / m$ and the phase increases by $\pi / 2$.
(Compare to previous slide)

## Saddle points

The saddle points are critical points for the phase function $\phi(z):=\log \left(z^{-m} f(z)\right)$. Recalling the logarithmic derivative of $f$,

$$
\phi^{\prime}(z)=-\frac{m}{z}+\sum_{x \in N} \frac{1}{z-x} .
$$

Substitute $y=z / m$, hoping that $y \approx \pm i / \pi$ will be a zero of:

$$
-\frac{1}{y}+\sum_{x \in N / m} \frac{1 / m}{y-x}
$$

The RHS is an integral against a Poisson process of points with density $m$, weighted at $1 / m$ each.
$\longrightarrow$ Lebesgue measure

## Wrapping up

The conditionally convergent symmetric integral

$$
\int \frac{d x}{y-x}
$$

is equal to $-i \pi$ in the UHP and $i \pi$ in the LHP (it does not converge on the real line). Therefore at $y \approx \pm i / \pi$,

$$
-\frac{1}{y}+\sum_{x \in N / m} \frac{1 / m}{y-x / m} \approx-\frac{1}{i / \pi}-i \pi=0
$$

as desired.

## End of Lecture 1

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