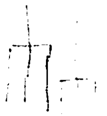
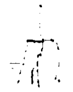
Hölder $\|g_1 g_2 g_3\|_1 \leq \|g_1\|_3 \|g_2\|_3 \|g_3\|_3$

$$\leq \sum_{x_1,x_2,x_3\in\mathbb{Z}^d} |f(x_1,x_2 x_3)| \quad \|p_t\|_3^3 \qquad \leq C \langle t\rangle^{-1-\delta}$$

all diagrams yield

to second order





$$\lambda^2 t \int dk_2 dk_3 dk_4\, \delta(k_1+k_2-k_3-k_4)\,\delta(\omega_1+\omega_2-\omega_3-\omega_4)\big(W_2W_3W_4 - W_1(W_2W_3 + W_2W_4 - W_3W_4)\big)$$
$$W_1W_2W_3W_4\left(\frac{1}{W_1}+\frac{1}{W_2}-\frac{1}{W_3}-\frac{1}{W_4}\right)$$

$$= \lambda^2 t\ \mathcal{C}(W)(k_1)$$

This leads to the

<u>Conjecture (kinetic limit).</u> The limit

$$\lim_{\lambda\to 0} W_\lambda(k,\lambda^{-2}t) = W(k,t).$$

exists and $W(k,t)$ is solution of

$$\partial_t W(k,t) = \mathcal{C}(W(t))(k) \qquad W(k,0) = W_0.$$

Lecture 8, Nov. 21

4.4 Did we find the right equation

- no momentum conservation

umklapp $k_1+k_2 = k_3+k_4 \mod 1$ Umklapp processes

- number and energy conservation

$$\frac{d}{dt}\int dk\, W(k,t) = \int dk_1\, \mathcal{C}(W(t))(k_1) = 0$$

$$\frac{d}{dt}\int dk\, \omega(k) W(k,t) = \int dk_1\, \omega(k_1)\, \mathcal{C}(W(t))(k_1)$$

$$= \int dk_1 dk_2 dk_3 dk_4\, \delta(\underline{k})\,\delta(\underline{\omega})\ W_1W_2W_3W_4 \underbrace{\overset{\text{odd}}{\left(\frac{1}{W_1}+\frac{1}{W_2}-\frac{1}{W_3}-\frac{1}{W_4}\right)}\omega_1}_{\frac{1}{4}(\omega_1+\omega_2-\omega_3-\omega_4)}$$

$$= 0$$

- the collisions are defined through the solution

$$\omega(k_1)+\omega(k_2) = \omega(k_3)+\omega(k_1+k_2-k_3)$$

can have many solution branches, numerics, counterintuitive

- entropy

finite volume Gauss measure

$$\frac{1}{Z} \prod_{x\in\Lambda} d\psi^*(x) d\psi(x) \; e^{-\frac{1}{2}\sum_{x,y\in\Lambda} h(x-y)\, \psi(x)^* \psi(y)} = \rho_{G,\Lambda}$$

entropy per unit volume

$$\frac{1}{|\Lambda|} S(\rho_{G,\Lambda}) = -\frac{1}{|\Lambda|} \int \rho_{G,\Lambda} \log \rho_{G,\Lambda} \prod_x d\psi d\psi^*$$

$$W(k) = \frac{1}{h}$$

$$\rightsquigarrow \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} S(\rho_{G,\Lambda}) = \int_{BZ} (\log W(k) \underbrace{+1}_{\text{normalization}}) dk$$

entropy production

$$\sigma(W) = \frac{d}{dt} \int \log W_1 \, dk_1 = \int \frac{1}{W_1} \frac{\partial}{\partial t} W_1 dk_1 = \int dk_1 .. dk_4 \;\delta(\underline{k})\; \delta(\underline{\omega})\, W_1 W_2 W_3 W_4 \left(\frac{1}{W_1} + \frac{1}{W_2} - \frac{1}{W_3} - \frac{1}{W_4}\right) \frac{1}{W_1}$$

$$= \frac{1}{4} \int dk_1\, dk_4 \;\delta(\underline{k})\, \delta(\underline{\omega})\, W_1 W_2 W_3 W_4 \left(\frac{1}{W_1} + \frac{1}{W_2} - \frac{1}{W_3} - \frac{1}{W_4}\right)^2 \geqslant 0$$

$$\sigma(W) = 0 \qquad \frac{1}{W_1} + \frac{1}{W_2} - \frac{1}{W_3} - \frac{1}{W_4} = 0$$

collision invariant $\quad \phi = \frac{1}{W}$

$$\phi_1 + \phi_2 - \phi_3 - \phi_4 = 0 \qquad \text{on the set} \quad \begin{matrix} k_1 + k_2 = k_3 + k_4 \\ \omega_1 + \omega_2 = \omega_3 + \omega_4 \end{matrix}$$

only solutions are $\quad \phi(k) = a + b\,\omega(k)$ $\quad$ *

$$W_{eq}(k) = \frac{\beta}{\underset{\text{chemical potential}}{-\mu} + \omega(k)} \qquad e^{-\beta\left(H_\Lambda - \mu \sum_{x\in\Lambda} |\psi(x)|^2\right)} \quad \text{thermal equilibrium}$$

$$\mu \leq \min_k \omega(k) \quad , \beta \geqslant 0$$

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* H. Spohn (collisional invariants)

Theorem $d \geqslant 2$. $\phi \in L^1(BZ)$, $\omega \in C^2(BZ)$, $\Lambda_{Hess} = \{ k \in BZ, \det Hess(\omega) = 0 \}$.

$\overline{\Lambda}_{Hess}$ is a set of codimension $\geqslant 1$.

arXiv 0605069

Then $\phi = a + b\omega$

$$\mathbb{E}_{eq}\left(|\psi(0)|^2 \right) = \beta \int_{\mathbb{T}^d} dk \, \frac{1}{\omega(k) - \mu}$$

$\|$ ASSUME $\omega(0) = 0$, $\omega(k) > 0$ for $k \neq 0$ $\|$

maximal $\mu = 0$

$$\rho_c = \beta \int dk \, \frac{1}{\omega(k)} \quad \begin{cases} = \infty & d = 1,2 \\ < \infty & d \geq 3 \end{cases}$$

where is the excess mass?

4.5 Bose-Einstein condensation

- static

$$\frac{1}{Z_\Lambda} e^{-\beta H_\Lambda} \, \delta(N_\Lambda - \rho |\Lambda|) \qquad N_\Lambda = \sum_{x \in \Lambda} |\psi(x)|^2$$

Berlin-Kac spherical model 1952, thermodynamics, $\rho > \rho_c$, non-Gaussian, 4th moment

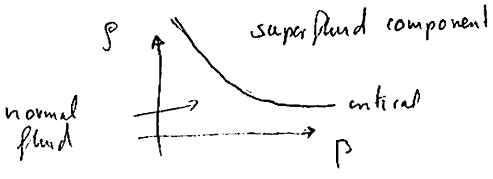
together with Jani, coupling, Wasserstein distance

$\rho \leq \rho_c$ $\quad \lim\limits_{\Lambda \to \mathbb{Z}^d} \psi_\Lambda(x) = \psi_\mu(x)$, ψ_μ Gaussian, covariance $\dfrac{\beta}{\omega(k) - \mu(\rho)}$

$\rho > \rho_c$ $\quad \lim\limits_{\Lambda \to \mathbb{Z}^d} \psi_\Lambda(x) = \psi_c(x) + (\rho - \rho_c)^{1/2} e^{i\vartheta}$

ψ_c Gaussian covariance $\dfrac{1}{\omega(k)}$, ϑ uniform on $[0, 2\pi]$.

critical



$\Rightarrow$ minimum at $k = k_0$ $\quad e^{i(2\pi k_0 \cdot x + \vartheta)}$

several minima, $k_1, , k_n$

$\sum\limits_{j=1}^{n} r_j e^{i(2\pi k_j \cdot x + \vartheta_j)}$, ϑ_j i.i.d , $\delta\left(\sum\limits_{j=1}^{n} r_j^2 - (\rho - \rho_c) \right) \prod r_j dr_j$

- dynamics

Wigner function $W(k) = \left(\underline{f(k)} + n \, \delta(k) \right) dk$

$\quad\quad$ fluid part $\quad\quad \uparrow$ superfluid classical analogue

non-zero fraction at velocity 0, rest normal, 0 velocity decays to 0

not for waves!!

coupled equations

$$\partial_t f_1 = \mathcal{C}(f)_1 + n\,\mathcal{D}(f)_1$$

$\mathcal{C}$ normal fluid

$\mathcal{D}$ set $W(k_j) = f(k_j) + n\delta(k_j)$

to linear order in n

$$\partial_t n = -n \int_{\mathbb{T}^d} dk_1\, \mathcal{D}(f)_1$$

ensures mass conservation

$$\partial_t (n + \int f) = n \int \mathcal{D}(f)_1 - n \int \mathcal{D}(f)_1 = 0$$

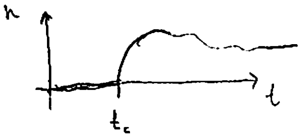
$\Rightarrow$ How is the condensate generated dynamically

initial value $f(k,0) = f_0(k)$, $\int_{\mathbb{T}^3} dk\, f_0(k) > \rho - \rho_c$, $n(0) = n$

- naive expectation

$n(t) = 0$ and $f(k,t) \to (\rho - \rho_c)\delta(k)$ as $t \to \infty$

- actual behavior



spontaneous generation

after blow-up fluid remains supercritical

Josserand, Pomeau, Rica, J. Low Temperature Physics 2006

Escobedo, Velazques, AMS monographs 2009

, Spohn arXiv: 0809.4551

for simplified model

$\mathbb{T}^3 \to \mathbb{R}^3$, $\omega(k) = k^2$, isotropic, $x = k^2$, $f(x)\,dx$

density $\sqrt{x} f(x)$

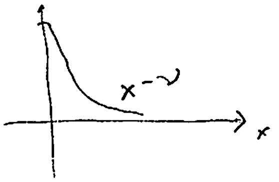
energy $x\sqrt{x} f(x)$

$x \geq 0$

δ-functions can be integrated explicitely

t_c time of blow-up

$$f(x, t_c - t) = \frac{1}{\tau(t)^{\nu}} \phi_-\left(\frac{x}{\tau(t)}\right) \quad \text{as } t \to 0_+$$



$\phi_-(x) = x^{-\nu}$, $\tau(t) = t^{1/(2\nu-1)}$

$\nu = 1.234$ (numerical)

$t > t_c$ $\quad f(x, t_c + t)$ $\quad \phi_+(x) = \frac{a_0(t)}{x}$, $x \to 0$,

plus more details

4.6 Bounds on the Duhamel expansion

initial equilibrium measure $\mathbb{E}_\lambda$ (not Gauss), not critical

$|t| \leq t_0$, dimension $d \geq 4$ for n.n. coupling ($|\nabla\psi|^2$)

<u>Theorem</u> (JL, HS, 2011)

$$\sum_{x\in\mathbb{Z}} e^{i2\pi k\cdot x}\, \mathbb{E}_\lambda\left(\psi^*(0,0)\,\psi(x,\lambda^{-2}t)\right) \xrightarrow[\lambda\to 0]{}$$ (integrated against test functions in x)

$$T(k) = T_1(k) + i\,T_2(k)$$

$$= e^{i\omega^\lambda \lambda^{-1} t}\; e^{-T_1(k)|t| - i\,T_2(k)\,t}$$

- $$T(k_1) = -2\int_0^\infty dt \int_{(\mathbb{T}^d)^3} dk_2\,dk_3\,dk_4\,\delta(\underline{k})\; e^{i(\omega_1+\omega_2-\omega_3-\omega_4)t}\,\left(W_3W_4 - W_2W_4 - W_2W_3\right)$$

"half linearized"

$$W(k) = \frac{1}{-\mu + \omega(k)} > 0$$

- $$\omega^\lambda = \omega(k) + \lambda\int dk\, W(k)$$

$\omega(k)$ ↑ trivial; $\lambda\int dk\,W(k)$ ↑ renormalized; independent of k!!

fast oscillations $\underbrace{e^{-i\lambda^{-1}(\int W(k)dk)\,t}}_{\text{cannot be expanded}}$

$\psi(x,t)$ Duhamel series up to N

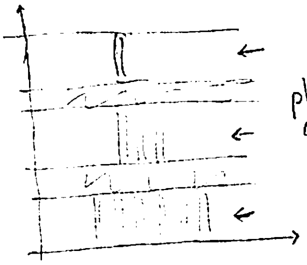
$$= \sum_{n=0}^{N} I_n(t) + I_{N+1}(t)$$

(i) $I_n(t) = I_n^{\text{lead}}(t) + I_n^{\text{rest}}(t)$

$I_n^{\text{lead}}(t) = a^n\,(\lambda^2 t)^n$ geometric

long-short diagrams

$(\lambda^2 t)^2$



$$|I_n^{\text{rest}}(t)| \leq (\lambda^2 t)^{n/2}\,\underset{\smile}{a}^n\; t^{-\delta} \qquad \delta > 0$$

estimate on equilibrium correlations — truncated

$$\sum_{\substack{\vec{x}\in\mathbb{Z}^n\\ x_1=0}} \left|\mathbb{E}_\lambda\Big(\prod_{i=1}^{n}\psi(x_j,\sigma_j)\Big)^T\right| \leq \lambda\,(c_0)^n\, n!$$

σ_j ↑ $*$ or not

Abdesselam, Procacci, Scoppola 2011

there are cancellations because of W_{eq}

only the expansion of $e^{-T(k)t}$ is different from 0.

$|I_n^{rel}(t)| \leq a^n (c_0)^n (\lambda^2 t)^n \; \lambda \; n! \; \lambda^{2\delta}$

cut-off

$N! \; \lambda^{\gamma} = 1 \qquad \gamma > 0$

in fact cut-off is soft.

polynomial of order 3^{N+1}

(ii) $I_{N+1}(t) = \mathbb{E}_\lambda \left(\psi(0,0)^* \int_0^t x_N(t,s)[\psi(s)] \right)$

$I_{N+1}(t)^2 \leq t \; \mathbb{E}_\lambda(|\psi(0,0)|^2) \int_0^t ds \; \mathbb{E}_\lambda\left(\left(\nu_N(t,s)[\psi(s)] \right)^2 \right)$

$s = 0$ by time stationarity

rescaled time required

$\sup_{0 \leq s \leq \lambda^{-2} t} \mathbb{E}_\lambda \left(\left(x_N(\lambda^{-2} t, s)[\psi(0)] \right)^2 \right) \leq \lambda^{4+\delta}$

- interlacing of $x_N(t,s)^2$