4. Nonlinear waves, wave turbulence

4.1 Hamiltonian models

prototype scalar field u (displacement)

$$\partial_t u = \Delta u - u - \lambda u^3 \quad , \quad u(x,t) \in \mathbb{R},\ x \in \mathbb{R}^d \quad , \quad \lambda \geq 0$$

- hamiltonian

$$H = \frac{1}{2}\int dx \left\{ \pi^2 + (\nabla_x u)^2 + u^2 + \frac{1}{2}\lambda u^4 \right\}$$

$$\dot{u} = \frac{\delta H}{\delta \pi} = \pi$$

$$\dot{\pi} = -\frac{\delta H}{\delta u} = \Delta u - u - \lambda u^3$$

- u scalar — usually $\vec{u}$ leads to elasticity, outside these lectures } more complicated story

- equilibrium measure $\lambda = 0$

Gaussian field π is white noise

spatial part Gaussian $\langle u(x)\,u(0)\rangle = \int dk\, e^{ikx} \frac{1}{k^2+1}$ | $d=3$: $\langle u(x)\,u(0)\rangle = \frac{1}{|x|} e^{-|x|}$ singular at short distances

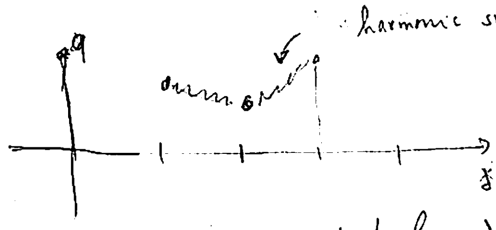
ultraviolet singular, extra difficulty

⇝ discrete models

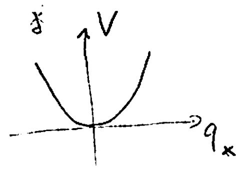
$\mathbb{R}^d \rightsquigarrow \mathbb{Z}^d$ $\quad \{q_j, p_j\} \in \mathbb{Z}^d$

$$H = \frac{1}{2}\sum_{j\in\mathbb{Z}^d} \left\{ p_j^2 + \sum_{\substack{e \\ |e|=1}} (q_{j+e} - q_j)^2 + q_j^2 + \frac{1}{2}\lambda q_j^4 \right\} \qquad \lambda \geq 0$$

harmonic spring



on-site potential

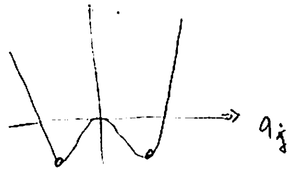


- physically for which the interaction should be translation invariant $V(q_{j+1} - q_j)$ "gradient models"

We consider effective one-band models

interaction is Ising like

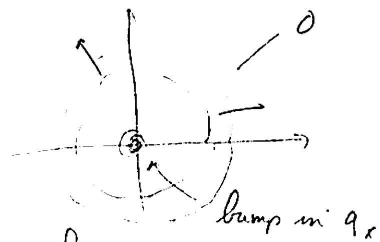
displacive phase transition



• We are far from the phase transition

$\lambda = 0$ discrete wave equation

$$\frac{d^2}{dt^2} q_j = (\Delta q)_j - q_j$$



+ dispersion from lattice

add small λ nonlinearity

physical one expects dissipation. How to arrive at a quantitative theory?

dissipation is a phenomenon at non-zero temperature

⇝ random initial data ⟸ + weak nonlinearity

prepare in Gibbs measure $\frac{1}{Z_\Lambda} e^{-\beta H_\Lambda(\underline{q},\underline{p})} \prod_{j\in\Lambda} dq_j \, dp_j$

perturb at $j=0$. How does the perturbation propagate?

*

We still have to switch to prototype

p and q enter asymmetricly, estimates simplify when q, p are on the same footing

⇝ discrete nonlinear Schrödinger equation (DNLS)

$x \in \mathbb{Z}^d$, wave field $\psi(x) \in \mathbb{C}$

$$i \frac{d}{dt} \psi(x) = \Delta \psi(x) - \lambda |\psi(x)|^2 \psi(x) \qquad \lambda \geq 0 \quad \text{(defocusing)}$$

* pioneering work of R. Peierls 1931 (post-doc with Pauli at ETH), correct answer, but crazy physical picture, still repeated in many text books.

Hamiltonian

$$H = \frac{1}{2}\sum_x \left\{ |\nabla\psi(x)|^2 + \lambda\frac{1}{2}|\psi(x)|^4 \right\}$$

canonical coordinates $\psi(x) = q_x + i p_x$

$$H = \frac{1}{2}\sum_x \left\{ |\nabla p_x|^2 + |\nabla q_x|^2 + \lambda\frac{1}{2}(p_x^2+q_x^2)^2 \right\}$$

$$\dot q_x = \partial_{p_x} H$$

$$\dot p_x = -\partial_{q_x} H$$

generalitation

rapid decay

$$H = \frac{1}{2}\sum_{x,y}\left\{ \underbrace{\alpha(x-y)\psi(y)^*\psi(x)}_{\text{hopping term}} + \lambda V(x-y)|\psi(x)|^2|\psi(y)|^2 \right\}$$

intensity at x times intensity at y

$V(x) = \delta_{0x}$ on-site

We stick to on-site.

infinite lattice (large box)

random initial data

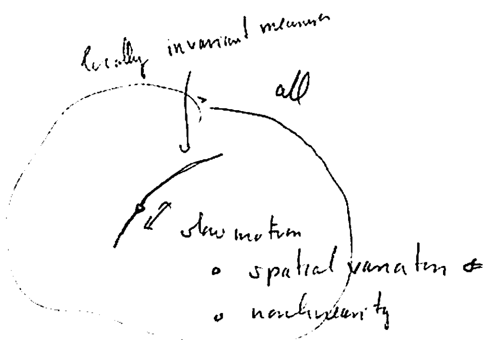
4.2 Wigner functions, semiclassical limit

Boltzmann f function suggests empirical measure for $\psi(x)$

no good oscillating

particles

local Poisson $\Leftarrow$ because stationary under free motion



|| stationary measures for $\lambda = 0$, translation invariant ||

$$\psi(x,t) = (e^{-i\Delta t}\psi_0)(x)$$

$$\psi(x,t)^* = (e^{i\Delta t}\psi_0^*)(x)$$

$\rightsquigarrow$ Gaussian measure

$$\langle\psi\rangle = 0$$

$$\langle\psi\psi\rangle = 0 \qquad \langle\psi^*\psi^*\rangle = 0$$

$$\langle\psi^*(x)\psi(y)\rangle = C(x-y) = \int_{\mathbb{T}^d} dk\, e^{i2\pi k\cdot(x-y)}\underbrace{W(k)}_{\text{Wigner function spatially homogeneous}}$$

Brioullin zone torus $[-\frac{1}{2},\frac{1}{2}]^d = \mathbb{T}^d$

$W \geq 0$, periodic, continuous

exhausts "all" regular, translation invariant measures (?)

see Dudnikova, Komech 2010 for discrete wave

in position space

$$\frac{1}{Z_\Lambda} e^{-\frac{1}{2}\sum_{x,y\in\Lambda} h(x-y)\, \psi^*(x)\psi(y)} \prod_{x\in\Lambda} d\psi^*(x)\, d\psi(x), \quad h(x) = h(-x)^*$$

$\hat{h}(k) > 0$ covariance $W = \frac{1}{\hat{h}}$

- slow variation

in spirit $h(x-y) \rightsquigarrow \tilde{h}(\varepsilon x, x-y)$

Definition: Wigner function (distribution, integrate against test functions)

$$W_\psi(x,p) = \sum_{y\in\mathbb{Z}^d} e^{i2\pi p\cdot y}\, \psi^*(x+\tfrac{1}{2}y)\,\psi(x-\tfrac{1}{2}y), \quad x\in(\tfrac{1}{2}\mathbb{Z})^d,\ p\in\mathbb{T}^d$$

see Teufel LNM 1821 (2003) for properties

scaled random field

$$\varepsilon^d \sum_x f(\varepsilon x)\, W_\psi(x,p) \qquad f \text{ rapid decay}$$

- law of large numbers

sequence of Gaussian measures μ_G^ε, covariance $\mathbb{E}_G^\varepsilon(\psi^*(x)\psi(y)) = C_\varepsilon(x,y)$

assume $|C_\varepsilon(x,y)| \leq h(x-y)$, $h\in\ell_1$

$$W_\varepsilon(x,p) = \sum_{y\in\mathbb{Z}^d} e^{i2\pi p\cdot y}\, C_\varepsilon(x+\tfrac{1}{2}y, x-\tfrac{1}{2}y)$$

assume $\lim_{\varepsilon\to 0} W_\varepsilon(\lfloor\varepsilon^{-1}r\rfloor, p) = W(r,p)$ continuous, $r\in\mathbb{R}^d$, $p\in\mathbb{T}$

Then

$$\lim_{\varepsilon\to 0} \varepsilon^d \sum_x f(\varepsilon x)\, W_\psi(x,p) = \int dr\, f(r)\, W(r,p) \qquad \text{(moments)}$$

- mean by definition

- covariance

$$\mathbb{E}_G^\varepsilon\big(\psi^*(x_1+\tfrac12 y_1)\psi(x_1-\tfrac12 y_1)\,\psi^*(x_2+\tfrac12 y_2)\psi(x_2-\tfrac12 y_2)\big)$$

$$= \mathbb{E}_G^\varepsilon(\ 1\ )\cdot\mathbb{E}_G^\varepsilon\ (2) \qquad\qquad \text{product}$$

$$+\ \mathbb{E}_G^\varepsilon\big(\psi^*(x_1+\tfrac12 y_1)\psi(x_2-\tfrac12 y_2)\big)\,\mathbb{E}_G^\varepsilon\big(\psi^*(x_2+\tfrac12 y_2)\psi(x_1-\tfrac12 y_1)\big)$$

scaling $\quad x_j \sim \lfloor \tfrac{1}{\varepsilon} r_j \rfloor \quad j=1,2 \quad$ widely separated

$$\Big|\mathbb{E}_G^\varepsilon\big(\psi^*(\lfloor\varepsilon^{-1}r_1\rfloor+\tfrac12 y_1)\,\psi(\lfloor\varepsilon^{-1}r_2\rfloor-\tfrac12 y_2)\big)\Big| \le h\big(\varepsilon^{-1}(r_1-r_2)+\tfrac12 y_1+\tfrac12 y_2\big)$$

$$\to 0 \quad \text{as } \varepsilon\to 0 \qquad \square$$

The random Wigner function W_ψ takes the role of the empirical measure for point processes.

dynamics

$$i\frac{d}{dt}\psi(x) = \sum_y \alpha(x-y)\,\psi(y) \qquad\qquad k \in \mathbb{T}^d$$

$$i\frac{d}{dt}\hat{\psi}(k) = \underbrace{\hat{\alpha}(k)}_{2\pi\,\omega(k)}\hat{\psi}(k) \quad \rightsquigarrow \quad \hat{\psi}(k,t) = e^{-i2\pi\omega(k)t}\,\hat{\psi}_0(k)$$

average Wigner function in Fourier space, scaled with lattice spacing ε

$$W(x,p) = \int_{\mathbb{T}^d} dk\, e^{i2\pi x\cdot k}\,\mathbb{E}_G^{\varepsilon}\left(\hat{\psi}^*(p-\tfrac{1}{2}k)\,\hat{\psi}(p+\tfrac{1}{2}k)\right)$$

$$W(\varepsilon^{-1}r, p, \varepsilon^{-1}t) = \int dk\, e^{i2\pi(r\cdot k)/\varepsilon}\, e^{i(\omega(p-\frac{k}{2})-\omega(p+\frac{k}{2}))\varepsilon^{-1}t}\,\hat{\psi}_0(p-\tfrac{k}{2})\,\hat{\psi}_0(p+\tfrac{k}{2})$$

ballistic

$$= \varepsilon^{+d}\int_{(\varepsilon^{-1}\mathbb{T})^d} dk\, e^{i2\pi rk}\, e^{-i\nabla\omega(p)\cdot k\, t}\,\mathbb{E}^{\varepsilon}\left(\hat{\psi}_0(p-\tfrac{\varepsilon k}{2})\,\hat{\psi}_0(p+\tfrac{\varepsilon k}{2})\right)$$

$$= \varepsilon^{d}\int_{(\varepsilon^{-1}\mathbb{T})^d} dk\, e^{i2\pi(r-\nabla\omega(p)t)\cdot k}\,\mathbb{E}^{\varepsilon}\left(\hat{\psi}_0(p-\tfrac{\varepsilon k}{2})\,\hat{\psi}(p+\tfrac{\varepsilon k}{2})\right)$$

back to position space — more explicit

$\rightsquigarrow$ under ballistic scaling $\lim_{\varepsilon\to 0} W(\varepsilon^{-1}r, p, \varepsilon^{-1}t) = W_0(r-\nabla\omega(p)t, p)$

initial Wigner

$$\left\| \partial_t W(r,p,t) = -\nabla\omega(p)\cdot\nabla_r W(r,p,t) \right\|$$

evolves as coming from particles with energy $E(p) = \omega(p)$ $\quad\|\quad \dot{q} = \nabla\omega(p)$, $\dot{p} = 0$

wave optics $\longrightarrow$ ray optics

semiclassical limit of Schrödinger equation

|| in the kinetic limit one has a particle picture ||

phonons, photons, ...