Lectures on Mean Field Games: I. The Two Pronged Probabilistic Approach & First Examples

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CREDITS

Joint Works with

François Delarue (Nice)

series of papers and two-volume book (forthcoming)

Colleagues and Ph.D. students

J.P. Fouque, A. Lachapelle, D. Lacker, P. Wang, G. Zhu

Agent Based Models and Mean Field Games

- **Agent Based Models** for large systems
  - Behavior prescribed at the individual (**microscopic**) level
  - Exogenously specified **interactions**
  - Large scale **simulations** possible
- If **symmetries** in the system, interactions can be **Mean Field**
  - Possible **averaging** effects for large populations
  - **Mean Field** limits easier to simulate and study
  - Net result: **Macroscopic** behavior of the system

Mean Field Games vs Agent Based Models

- **Mean Field Games**
  - At the (**microscopic**) level individuals control their states
  - **Exogenously** specified interaction rules
  - Individuals are rational: they **OPTIMIZE** !!!!
- Search for **equilibria**: very difficult, NP hard in general
- If **symmetries** in the system, interactions can be **Mean Field**
  - Possible **averaging** effects for large populations
  - **Mean Field** limits easier to study
  - **Macroscopic** behavior of the system thru solutions of

    Mean Field Games

    Lasry - Lions (MFG) **Caines - Huang - Malhamé** (NCE)
  - **Examples:** flocking, schooling, herding, crowd behavior, percolation of information, price formation, hacker behavior and cyber security, ...

A Few Examples

Example I: A Model of "Flocking"

Deterministic dynamical system model (**Cucker-Smale**)

$$\begin{cases} dx_t^i &= v_t^i dt \\ dv_t^i &= \frac{1}{N}\sum_{j=1}^N w_{i,j}(t)[v_t^i - v_t^j]dt \end{cases}$$

for the weights

$$w_{i,j}(t) = w(|x_t^i - x_t^j|) = \frac{\kappa}{(1+|x_t^i - x_t^j|^2)^\beta}$$

for some $\kappa > 0$ and $\beta \geq 0$.

If N fixed, $0 \leq \beta \leq 1/2$

- $\lim_{t\to\infty} v_t^i = \overline{v}_0^N$, for $i = 1, \cdots, N$
- $\sup_{t\geq 0} \max_{i,j=1,\cdots,N} |x_t^i - x_t^j| < \infty$

Many extensions/refinements since original C-S contribution.

A MFG FORMULATION

(**Nourian-Caines-Malhamé**)

$X_t^i = [x_t^i, v_t^i]$ state of player i

$$\begin{cases} dx_t^i & = v_t^i dt \\ dv_t^i & = \alpha_t^i dt + \sigma dW_t^i \end{cases}$$

For strategy profile $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N)$, the cost to player i

$$J^i(\boldsymbol{\alpha}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\frac{1}{2}|\alpha_t^i|^2 + \frac{1}{2}\left| \frac{1}{N} \sum_{j=1}^N w(|x_t^i - x_t^j|)[v_t^i - v_t^j] \right|^2 \right) dt$$

- **Ergodic** (infinite horizon) model;
- $\beta = 0$, **Linear Quadratic** (LQ) model;
- if $\beta > 0$, **asymptotic expansions** for $\beta << 1$?

Reformulation

$$J^i(\boldsymbol{\alpha}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f^i(t, X_t, \overline{\mu}_t^N, \alpha_t)dt$$

with

$$f^i(t, X, \mu, \alpha) = \frac{1}{2}|\alpha^i|^2 + \frac{1}{2}\left| \int w(|x - x'|)[v - v']\mu(dx', dv')\right|^2$$

where $\alpha = (\alpha^1, \cdots, \alpha^N)$, $X = [x, v]$, and $X' = [x, v]$.

Unfortunately

f^i is not convex !

EXAMPLE II: CONGESTION & FORCED EXIT

Lasry-Lions-Achdou- ....

- bounded domain D in $\mathbb{R}^d$
- exit only possible through $\Gamma \subset \partial D$

$$dX_t^i = \alpha_t^i dt + dW_t^i + dK_t^i, \quad t \in [0, T], \ X_0^i = x_0^i \in D$$

- **reflecting** boundary conditions on $\partial D \setminus \Gamma$
- **Dirichlet** boundary condition on Γ

$$J^i(\boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N) = \mathbb{E}\Big[\int_0^{T\wedge\tau^i} \Big(\frac{1}{2}\ell(X_t^i, \mu_t^N)|\alpha_t|^2 + f(t)\Big)dt\Big]$$

- f **penalizes** the time spent in D before the exit
- $\ell(x, \mu)$ models **congestion** around x if μ is the distribution of the individuals (e.g. $\ell(x, \mu) = m(x)^\alpha$)

Congestion & Exit of a Room

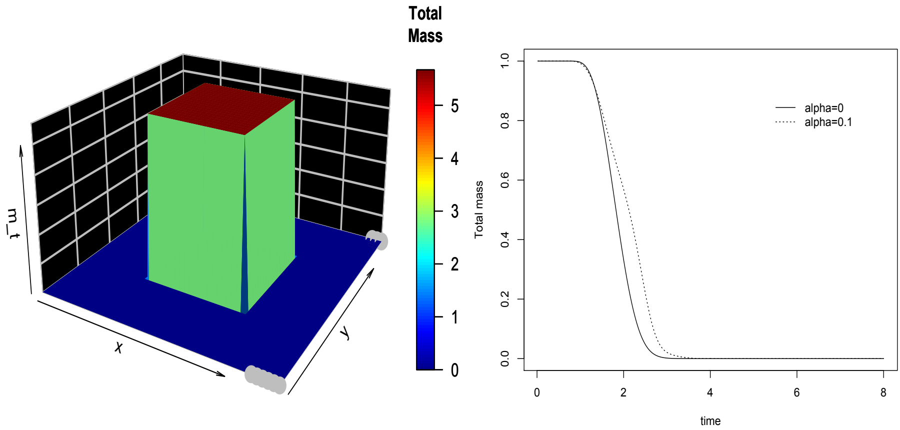
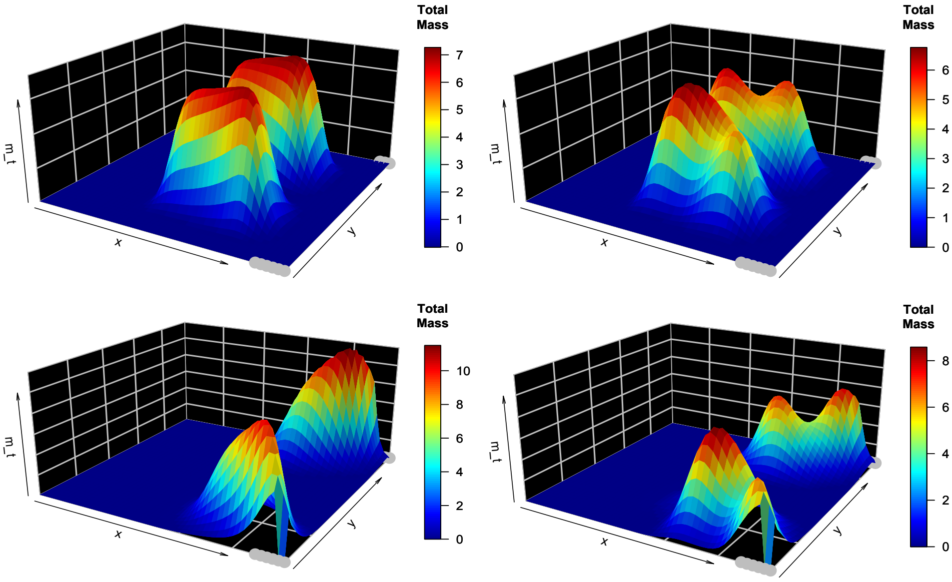
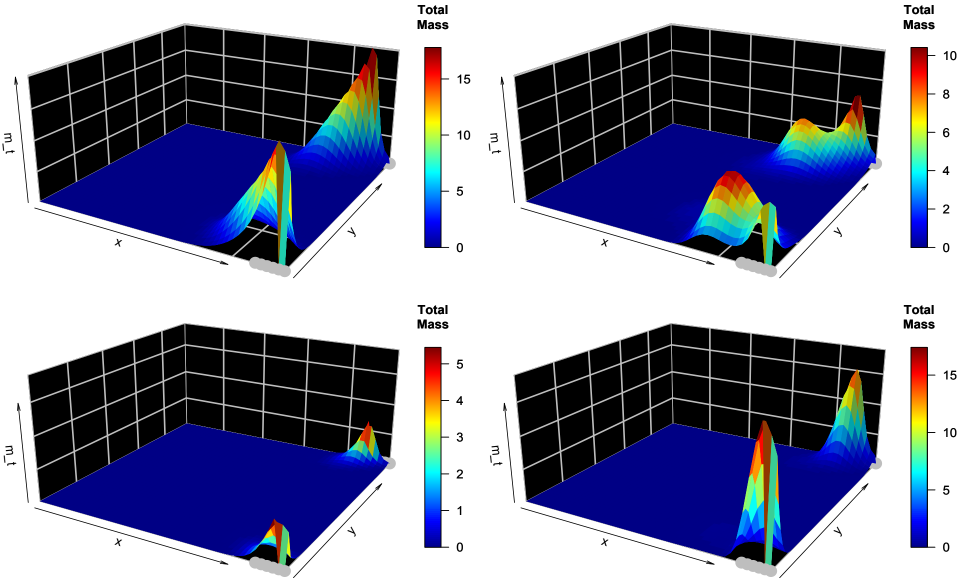


Figure: Left: Initial distribution m_0. Right: Time evolution of the total mass of the distribution m_t of the individuals still in the room at time t without congestion (continuous line) and with moderate congestion (dotted line).

Room Exit Densities



Room Exit Densities



Example III: Toy Model for Systemic Risk

R.C. + J.P. Fouque

- $X_t^i, i = 1, \ldots, N$ log-monetary reserves of N banks
- $W_t^i, i = 0, 1, \ldots, N$ **independent Brownian motions**, $\sigma > 0$
- **Borrowing, lending, and re-payments** through the drifts:

$$dX_t^i = \left[\alpha_t^i - \alpha_{t-\tau}^i\right] dt + \sigma\left(\sqrt{1-\rho^2}\,dW_t^i + \rho\, dW_t^0\right), \quad i = 1, \cdots, N$$

α^i **is the control of bank** i which tries to **minimize**

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left\{\int_0^T \left[\frac{1}{2}(\alpha_t^i)^2 - q\alpha_t^i(\overline{X}_t - X_t^i) + \frac{\epsilon}{2}(\overline{X}_t - X_t^i)^2\right] dt + \frac{\epsilon}{2}(\overline{X}_T - X_T^i)^2\right\}$$

Regulator chooses $q > 0$ to control the cost of borrowing and lending.

- If X_t^i is small (relative to the empirical mean $\overline{X}_t$) then bank i will want to borrow($\alpha_t^i > 0$)
- If X_t^i is large then bank i will want to lend ($\alpha_t^i < 0$)

Example of **Mean Field Game (MFG)** with a **common noise** W^0 and **delay** in the controls. **No delay in these lectures !**

- Interesting features
  - Multi-period (continuous time) **dynamic equilibrium** model
  - **Explicitly solvable** (without delay !)
    - in **open loop** form
    - in **closed loop** form
    - solutions are **different for N finite** !
- Shortcomings
  - **Naive** model of bank lending, borrowing, and re-payments
  - Only a small jab at **stability** of the system
- Challenging Extensions:
  - Introduction of **major** and **minor** players
  - Better solutions & understanding of **time delays**
  - Introduction of **constraints**

Example IV: Price Impact of Traders

X_t^i number of shares owned at time t, α_t^i *rate of trading*

$$dX_t^i = \alpha_t^i\, dt + \sigma^i dW_t^i$$

K_t^i amount of cash held by trader i at time t

$$dK_t^i = -[\alpha_t^i S_t + c(\alpha_t^i)]\, dt,$$

where S_t price of one share, $\alpha \to c(\alpha) \geq 0$ *cost for trading at rate* α
Price impact formula:

$$dS_t = \frac{1}{N} \sum_{i=1}^{N} h(\alpha_t^i)\, dt + \sigma_0 dW_t^0$$

Trader i tries to minimize

$$J^i(\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^N) = \mathbb{E}\left[\int_0^T c_X(X_t^i) dt + g(X_T^i) - V_T^i \right]$$

where V_t^i is the wealth of trader i at time t:

$$V_t^i = K_t^i + X_t^i S_t.$$

Example of an **Extended Mean Field Game**

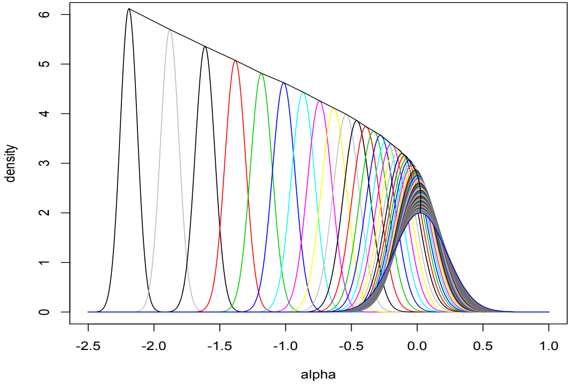


FIGURE: Time evolution (from t ranging from 0.06 to $T = 1$) of the marginal density of the optimal rate of trading $\hat{\alpha}_t$ for a representative trader.

Terminal Inventory of a Typical Trader

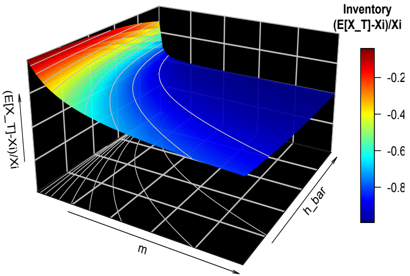


FIGURE: Expected terminal inventory as a function of m and c_X (left) , and as a function of m and $\overline{h}$ (right).

TERMINAL INVENTORY OF A TYPICAL TRADER

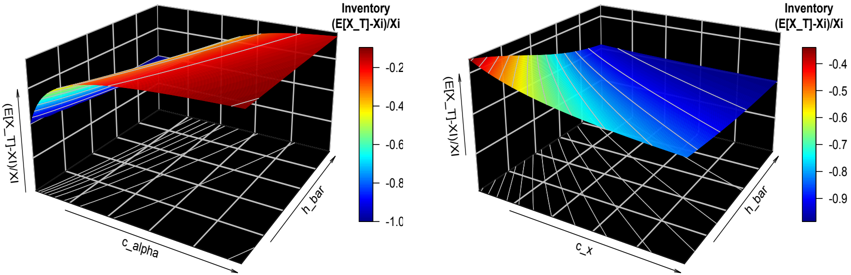


FIGURE: Expected terminal inventory as a function of c_α and $\overline{h}$ (left), and as a function of c_X and $\overline{h}$ (right).

Example V: Macro - Economic Growth Model

Krusell - Smith in **Aiyagari**'s diffusion form

- Z_t^i labor productivity of worker $i \in \{1, \cdots, N\}$
- A_t^i wealth at time t
- $\sigma_Z(\cdot)$ and $\mu_Z(\cdot)$ given functions

$$\begin{cases} dZ_t^i & = \mu_Z(Z_t^i)dt + \sigma_Z(Z_t^i)dW_t^i \\ dA_t^i & = [w_t^i Z_t^i + r_t A_t^i - c_t^i]dt, \end{cases}$$

- r_t **interest rate**, w_t^i **wages** of worker i at time t
- c_t^i **consumption** (**control**) of worker i

In a competitive equilibrium

$$\begin{cases} r_t & = [\partial_K F](K_t, L_t)|_{L_t=1} - \delta \\ w_t & = [\partial_L F](K_t, L_t)|_{L_t=1} \end{cases}$$

where $(K, L) \mapsto F(K, L)$ production function and

$$K_t = \int a\overline{\mu}^N_{X_t}(dz, da) = \frac{1}{N}\sum_{i=1}^N A_t^i,$$

Mean Field Interaction

Example V (cont.)l

Optimization Problem

$$\max \quad J^i(\boldsymbol{c}^1, \cdots, \boldsymbol{c}^N) = \mathbb{E}\int_0^\infty e^{-\rho t} U(c_t^i)dt, \quad i = 1, \cdots, N$$

with CRRA isoelastic **utility function**

$$U(c) = \frac{c^{1-\gamma} - 1}{1-\gamma},$$

Cobb - Douglas production function

$$F(K, L) = \bar{a}\, K^\alpha\, L^{1-\alpha},$$

for some constants $a > 0$ and $\alpha \in (0, 1)$ so in equilibrium:

$$r_t = \alpha\bar{a}K_t^{\alpha-1}L_t^{1-\alpha} - \delta, \qquad \text{and} \qquad w_t = (1-\alpha)\bar{a}K_t^\alpha L_t^{-\alpha},$$

Normalize the aggregate supply of labor to $L_t \equiv 1$,

$$r_t = \frac{\alpha\bar{a}}{K_t^{1-\alpha}} - \delta, \qquad \text{and} \qquad w_t = (1-\alpha)\bar{a}K_t^\alpha,$$

Singular coefficients !

EXAMPLE VI: FINITE STATE SPACES

Cyber Security (**Bensoussan - Kolokolstov**)

- Finite state space $E = \{1, \cdots, M\}$,
- Markovian models.
- Dynamics given by Q-matrices (rates of jump)
- Controls given by feedback functions of the current state.

Example VII: Games with Major and Minor Players

Examples:

- Financial system
  - Finite (small) number of SIFIs
  - Large number of small banks
- Population Biology (Bee swarming)
  - Finite (small) number of streakers
  - Large number of worker bees
- Economic Contract Theory
  - Regulator proposing a contract
  - Utilities operating under the regulation
  - Open Question: Nash versus Stackelberg

Ex. VIII: Games with Major and Minor Players

$$\begin{cases} dX_t^0 &= b_0(t, X_t^0, \overline{\mu}_t, \alpha_t^0)dt + \sigma_0(t, X_t^0, \overline{\mu}_t, \alpha_t^0)dW_t^0 \\ dX_t^i &= b(t, X_t^i, \overline{\mu}_t, X_t^0, \alpha_t^i, \alpha_t^0)dt + \sigma(t, X_t^i, \overline{\mu}_t, X_t^0, \alpha_t^i, \alpha_t^0 dW_t, \qquad i = 1, \cdots, N \end{cases}$$

where $\overline{\mu}_t^N$ is the empirical distribution of $X_t^1, \cdots, X_t^N$.

Cost functionals:

$$\begin{cases} J^0(\boldsymbol{\alpha}^0, \boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N) &= \mathbb{E}\big[\int_0^T f_0(t, X_t^0, \overline{\mu}_t, \alpha_t^0)dt + g^0(X_T^0, \overline{\mu}_T)\big] \\ J^(\boldsymbol{\alpha}^0, \boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N) &= \mathbb{E}\big[\int_0^T f(t, X_t^i, \overline{\mu}_t^N, X_t^0, \alpha_t^i, \alpha_t^0)dt + g(X_T^i, \overline{\mu}_T)\big], \qquad i = 1, \cdots, N \end{cases}$$

EXAMPLE IX: MEAN FIELD GAMES OF TIMING

Last Lecture.

- Illiquidity Modeling and Bank Runs
- Modeling the large issuance of a convertible bond

The Mean Field Game Strategy & the Mean Field Game Problem

Classical Stochastic Differential Control

$$\inf_{\boldsymbol{\alpha}\in\mathbb{A}} \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t)dt + g(X_T, \mu_T)\right]$$

$$\text{subject to} \quad dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dW_t; \quad X_0 = x_0.$$

- **Analytic Approach** (by PDEs)
  - HJB equation
- **Probabilistic Approaches** (by FBSDEs)
  1. Represent value function as solution of a BSDE
  2. Represent the gradient of the value function as solution of a FBSDE (Stochastic Maximum Principle)

I. First Probabilistic Approach

Assumptions

- σ is uncontrolled
- σ is invertible

Reduced **Hamitonian**

$$H(t,x,y,\alpha) = b(t,x,\alpha)\cdot y + f(t,x,\alpha)$$

For each control $\boldsymbol{\alpha}$ solve **BSDE**

$$dY_t^{\boldsymbol{\alpha}} = -H(t,X_t,Z_t\sigma(t,X_t)^{-1},\alpha_t)dt + Z_t\cdot dW_t, \qquad Y_T^{\boldsymbol{\alpha}} = g(X_T)$$

Then

$$Y_0^{\boldsymbol{\alpha}} = J(\boldsymbol{\alpha}) = \mathbb{E}\left[\int_0^T f(t,X_t,\alpha_t)dt + g(X_T,\mu_T)\right]$$

So by **comparison theorems** for BSDEs, optimal control $\hat{\boldsymbol{\alpha}}$ given by:

$$\hat{\alpha}_t = \hat{\alpha}(t,X_t,Z_t\sigma(t,X_t)^{-1}), \quad \text{with} \quad \hat{\alpha}(t,x,y)\in \text{argmin}_{\alpha\in A}H(t,x,y,\alpha)$$

and $Y_0^{\boldsymbol{\alpha}} = J(\hat{\boldsymbol{\alpha}})$

II. Pontryagin Stochastic Maximum Approach

Assumptions

- Coefficients b, σ and f differentiable
- f convex in (x, α) and g convex

Hamitonian

$$H(t, x, y, z, \alpha) = b(t, x, \alpha) \cdot y + \sigma(t, x, \alpha) \cdot z + f(t, x, \alpha)$$

For each control $\boldsymbol{\alpha}$ solve **BSDE** for the adjoint processes $\mathbf{Y} = (Y_t)_t$ and $\mathbf{Z} = (Z_t)_t$

$$dY_t = -\partial_x H(t, X_t, Y_t, Z_t, \alpha_t)dt + Z_t \cdot dW_t, \qquad Y_T = \partial_x g(X_T)$$

Then, optimal control $\hat{\boldsymbol{\alpha}}$ given by:

$$\hat{\alpha}_t = \hat{\alpha}(t, X_t, Y_t, Z_t), \quad \text{with} \quad \hat{\alpha}(t, x, y, z) \in \text{argmin}_{\alpha \in A} H(t, x, y, z, \alpha)$$

Summary

In both cases (σ uncontrolled), need to **solve a FBSDE**

$$\begin{cases} dX_t = B(t, X_t, Y_t, Z_t)dt + \Sigma(t, X_t)dW_t, \\ dY_t = -F(t, X_t, Y_t, Z_t)dt + Z_t dW_t \end{cases}$$

First Approach

$$\begin{aligned} B(t,x,y,z) &= b\big(t,x,\hat{\alpha}(t,x,z\sigma(t,x)^{-1})\big), \\ F(t,x,y,z) &= -f\big(t,x,\hat{\alpha}(t,x,z\sigma(t,x)^{-1}\big) \\ &\qquad - \big(z\sigma(t,x,)^{-1}\big)\cdot b\big(t,x,\hat{\alpha}(t,x,z\sigma(t,x)^{-1})\big). \end{aligned}$$

Second Approach

$$\begin{aligned} B(t,x,y,z) &= b(t,x,\hat{\alpha}(t,x,y)), \\ F(t,x,y,z) &= -\partial_x f(t,x,\hat{\alpha}(t,x,y)) - y\cdot\partial_x b(t,x,\hat{\alpha}(t,x,y)). \end{aligned}$$

FBSDE Decoupling Field

To solve the **standard FBSDE**

$$\begin{cases} dX_t = B(t, X_t, Y_t)dt + \Sigma(t, X_t)dW_t \\ dY_t = -F(t, X_t, Y_t)dt + Z_t dW_t \end{cases}$$

with $X_0 = x_0$ and $Y^T = g(X_T)$,

a standard approach is to look for a solution of the form $Y_t = u(t, X_t)$

- $(t, x) \hookrightarrow u(t, x)$ is called the **decoupling field** of the FBSDE
- If u is smooth,
  - apply Itô's formula to $du(t, X_t)$ using forward equation
  - identify the result with dY_t in backward equation

  $(t, x) \hookrightarrow u(t, x)$ is the solution of a nonlinear PDE

Oh well, So much for the probabilistic approach !

Propagation of Chaos & McKean-Vlasov SDEs

System of N particles $X_t^{N,i}$ at time t with **symmetric (Mean Field)** interactions

$$dX_t^{N,i} = b(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N})dt + \sigma(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N})dW_t^i, \quad i = 1, \cdots, N$$

where $\overline{\mu}^N_{X_t^N}$ is the empirical measure $\overline{\mu}^N_{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$

Large population asymptotics ($N \to \infty$)

1. The N processes $\mathbf{X}^{N,i} = (X_t^{N,i})_{0 \le t \le T}$ become asymptotically **i.i.d.**
2. Each of them is (asymptotically) distributed as the solution of the McKean-Vlasov SDE

$$dX_t = b(t, X_t, \mathcal{L}(X_t))dt + \sigma(t, X_t, \mathcal{L}(X_t))dW_t$$

Frequently used notation:

$$\mathcal{L}(X) = \mathbb{P}_X \qquad \text{distribution of the random variable } X.$$

Forward SDEs of McKean-Vlasov Type

$$dX_t = B\big(t, X_t, \mathcal{L}(X_t)\big)\,dt + \Sigma\big(t, X_t, \mathcal{L}(X_t)\big)\,dW_t, \qquad T \in [0, T].$$

Assumption. There exists a constant $c \geq 0$ such that

(A1) For each $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, the processes $B(\cdot, \cdot, x, \mu) : \Omega \times [0, T] \ni (\omega, t) \mapsto B(\omega, t, x, \mu)$ and $\Sigma(\cdot, \cdot, x, \mu) : \Omega \times [0, T] \ni (\omega, t) \mapsto \Sigma(\omega, t, x, \mu)$ are $\mathbb{F}$-progressively measurable and belong to $\mathbb{H}^{2,d}$ and $\mathbb{H}^{2,d\times d}$ respectively.

(A2) $\forall t \in [0, T]$, $\forall x, x' \in \mathbb{R}^d$, $\forall \mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, with probability 1 under $\mathbb{P}$,

$$|B(t, x, \mu) - B(t, x', \mu')| + |\Sigma(t, x, \mu) - \Sigma(t, x', \mu')| \leq c\big[|x - x'| + W_2(\mu, \mu')\big],$$

where W_2 denotes the 2-Wasserstein distance on the space $\mathcal{P}_2(\mathbb{R}^d)$.

Result. if $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$, then there exists a unique solution $\mathbf{X} = (X_t)_{0\leq t\leq T}$ in $\mathbb{S}^{2,d}$ s.t. for every $p \in [1, 2]$

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |X_t|^p\Big] < +\infty.$$

Sznitmann

N-Player Stochastic Differential Games

Assume Mean Field Interactions (symmetric game)

$$dX_t^{N,i} = b(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N}, \alpha_t^i)dt + \sigma(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N}, \alpha_t^i)dW_t^i \quad i = 1, \cdots, N$$

Assume player i tries to minimize

$$J^i(\boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N) = \mathbb{E}\left[\int_0^T f(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N}, \alpha_t^i)dt + g(X_T, \overline{\mu}^N_{X_T^N})\right]$$

Search for Nash equilibria

- Very difficult in general, even if N is small
- ϵ-Nash equilibria? Still hard.
- How about in the limit $N \to \infty$?

Mean Field Games Lasry - Lions, Caines-Huang-Malhamé

MFG PARADIGM

A **typical** agent plays against a **field** of players whose states he/she feels through the statistical distribution **distribution** μ_t of their states at time t

1. For each **Fixed** measure flow $\boldsymbol{\mu} = (\mu_t)$ in $\mathcal{P}(\mathbb{R})$, solve the **standard stochastic control problem**

$$\hat{\boldsymbol{\alpha}} = \arg \inf_{\boldsymbol{\alpha} \in \mathbb{A}} \mathbb{E} \left\{ \int_0^T f(t, X_t, \mu_t, \alpha_t) dt + g(X_T, \mu_T) \right\}$$

subject to

$$dX_t = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t$$

2. **Fixed Point Problem**: determine $\boldsymbol{\mu} = (\mu_t)$ so that

$$\forall t \in [0, T], \quad \mathcal{L}(X_t^{\hat{\boldsymbol{\alpha}}}) = \mu_t.$$

$\boldsymbol{\mu}$ or $\hat{\boldsymbol{\alpha}}$ is called a solution of the MFG.

Once this is done one expects that, if $\hat{\alpha}_t = \phi(t, X_t)$,

$$\alpha_t^{j*} = \phi^*(t, X_t^j), \qquad j = 1, \cdots, N$$

form an **approximate Nash equilibrium** for the game with N players.

Solving MFGs by Solving FBSDEs of McKean-Vlasov Type

Minimization of the (Reduced) Hamiltonian

Recall

$$H(t, x, \mu, y, \alpha) = y \cdot b(t, x, \mu, \alpha) + f(t, x, \mu, \alpha)$$

and we want to use

$$\hat{\alpha}(t, x, \mu, y) \in \arg \in_{\alpha \in A} H(t, x, \mu, y, \alpha).$$

(A.1) b is affine in α: $b(t, x, \mu, \alpha) = b_1(t, x, \mu) + b_2(t)\alpha$ with b_1 and b_2 bounded.
(A.2) Running cost f uniformly λ-convex for some $\lambda > 0$:

$$f(t, x', \mu, \alpha') - f(t, x, \mu, \alpha) - \langle (x' - x, \alpha' - \alpha), \partial_{(x,\alpha)} f(t, x, \mu, \alpha) \rangle \geq \lambda |\alpha' - \alpha|^2.$$

Then

$\hat{\alpha}(t, x, \mu, y)$ ***is unique and***

$$[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (t, x, \mu, y) \to \hat{\alpha}(t, x, \mu, y)$$

is measurable, locally bounded and Lipschitz-continuous with respect to (x, y)***, uniformly in*** $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$

I. Value Function Representation

Recall

$$\sigma(t,x,\mu,\alpha)=\sigma(t,x) \text{ **uniformly Lip-1 and uniformly elliptic**}$$

- If $A \subset \mathbb{R}^k$ is **bounded** (not really needed),
- if $\mathbf{X}^{t,x}=(X_s^{t,x})_{t\le s\le T}$ is the unique strong solution of $dX_t=\sigma(t,X_t)dW_t$ over $[t,T]$ s.t. $X_t^{t,x}=x$,
- if $(\hat{\boldsymbol{Y}}^{t,x},\hat{\boldsymbol{Z}}^{t,x})$ is a solution of the BSDE
$$d\hat{Y}_s^{t,x}=-H(t,X_s^{t,x},\mu_s,\hat{Z}_s^{t,x}\sigma(s,X_s^{t,x})^{-1},\hat{\alpha}(s,X_s^{t,x},\mu_s,\hat{Z}_s^{t,x}\sigma(s,X_s^{t,x})^{-1}))ds-\hat{Z}_s^{t,x}dW_s,$$
for $t\le s\le T$ with $\hat{Y}_T=g(X_T^{t,x},\mu_T)$,

then
$$\hat{\alpha}_t=\hat{\alpha}(s,X_s^{t,x},\mu_s,\hat{Z}_s^{t,x}\sigma(s,X_s^{t,x})^{-1})$$
is an optimal control over the interval $[t,T]$ and the value of the problem is given by:
$$V(t,x)=\hat{Y}_t^{t,x}.$$

The value function appears as the decoupling field of an FBSDE.

Fixed Point Step $\Longrightarrow$ McKean-Vlasov FBSDE

Starting from $t = 0$ and dropping the superscript ^{t,x}, for each fixed flow $\boldsymbol{\mu} = (\mu_t)_{0 \le t \le T}$

$$\begin{cases} dX_t = b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, Z_t\sigma(t, X_t)^{-1}))dt + \sigma(t, X_t)dW_t \\ dY_t = -H(t, X_t, \mu_t, Z_t\sigma(t, X_t)^{-1}, \hat{\alpha}(t, X_t, \mu_t, Z_t\sigma(t, X_t)^{-1}))dt - Z_t dW_t, \end{cases}$$

for $0 \le t \le T$, with $\hat{Y}_T = g(X_T, \mu_T)$.

Implementing the **fixed point step**

$$\mu_t \qquad \hookrightarrow \qquad \mathcal{L}(X_t)$$

gives an **FBSDE of McKean-Vlasov** type !

Again, **freeze** $\boldsymbol{\mu} = (\mu_t)_{0 \le t \le T}$,

Recall (reduced) Hamiltonian

$$H(t, x, \mu, y, \alpha) = b(t, x, \mu, \alpha) \cdot y + f(t, x, \mu, \alpha)$$

Adjoint processes

Given an admissible control $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$ and the corresponding controlled state process $\mathbf{X}^\alpha = (X_t^\alpha)_{0 \le t \le T}$, any couple $(Y_t, Z_t)_{0 \le t \le T}$ satisfying:

$$\begin{cases} dY_t = -\partial_x H(t, X_t^\alpha, \mu_t, Y_t, \alpha_t)dt + Z_t dW_t \\ Y_T = \partial_x g(X_T^\alpha, \mu_T) \end{cases}$$

is called a set of **adjoint processes**.

Stochastic Control Step

Use

$$\hat{\alpha}(t, x, \mu, y) = \arg\inf_{\alpha} H(t, x, \mu, y, \alpha),$$

inject it in **FORWARD** and **BACKWARD** dynamics and **SOLVE**

$$\begin{cases} dX_t = b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + \sigma(t, X_t)dW_t \\ dY_t = -\partial_x H(t, X, \mu_t, Y_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + Z_t dW_t \end{cases}$$

with $X_0 = x_0$ and $Y_T = \partial_x g(X_T, \mu_T)$

Standard **FBSDE** (for each **fixed** $t \hookrightarrow \mu_t$)

Fixed Point Step

Solve the **fixed point problem**

$$\boldsymbol{\mu} = (\mu_t)_{0\le t\le T} \quad \longrightarrow \quad \mathbf{X} = (X_t)_{0\le t\le T} \quad \longrightarrow \quad (\mathcal{L}(X_t))_{0\le t\le T}$$

Note: if we enforce $\mu_t = \mathcal{L}(X_t)$ for all $0 \le t \le T$ in FBSDE we have

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t), \hat{\alpha}(t, X_t, \mathcal{L}(X_t), Y_t))dt + \sigma(t, X_t)dW_t, \\ dY_t = -\partial_x H(t, X_t^\alpha, \mathcal{L}(X_t), Y_t, \hat{\alpha}(t, X_t, \mathcal{L}(X_t), Y_t))dt + Z_t dW_t \end{cases}$$

with

$$X_0 = x_0 \qquad \text{and} \qquad Y_T = \partial_x g(X_T, \mathcal{L}(X_T))$$

FBSDE of McKean-Vlasov type !!!

Very difficult

FBSDEs of McKean - Vlasov Type

In both probabilistic approaches to the MFG problem the problem reduces to the solution of an FBSDE

$$\begin{cases} dX_t = B(t, X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + \Sigma(t, X_t, \mathcal{L}(X_t))dW_t, \\ dY_t = -F(t, X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + Z_t dW_t \end{cases}$$

with, in the first approach

$$\begin{cases} B(t, x, \mu, y, z) = b(t, x, \mu, \hat{\alpha}(t, x, \mu, z\sigma(t, x)^{-1})), \\ F(t, x, \mu, y, z) = -f(t, x, \mu, \hat{\alpha}(t, x, \mu, z\sigma(t, x)^{-1}) \\ \qquad\qquad\qquad -z\sigma(t, x)^{-1} b(t, x, \mu, \hat{\alpha}(t, x, \mu, z\sigma(t, x)^{-1})), \end{cases}$$

and in the second:

$$\begin{cases} B(t, x, \mu, y, z) = b(t, x, \mu, \hat{\alpha}(t, x, \mu, y)), \\ F(t, x, \mu, y, z) = -\partial_x f(t, x, \mu, \hat{\alpha}(t, x, \mu, y)) - y\partial_x b(t, x, \mu, \hat{\alpha}(t, x, \mu, y)). \end{cases}$$

A Typical Existence Result

We try to solve:

$$\begin{cases} dX_t = B\big(t, X_t, Y_t, Z_t, \mathbb{P}_{(X_t,Y_t)}\big)\,dt + \Sigma\big(t, X_t, Y_t, \mathbb{P}_{(X_t,Y_t)}\big)\,dW_t \\ dY_t = -F\big(t, X_t, Y_t, Z_t, \mathbb{P}_{(X_t,Y_t)}\big)\,dt + Z_t dW_t, \quad 0 \le t \le T, \end{cases}$$

with $X_0 = x_0$ and $Y_T = G(X_T, \mathbb{P}_{X_T})$.

Assumptions

(A1). B, F, Σ and G are continuous in μ and uniformly (in μ) Lipschitz in (x, y, z)

(A2). Σ and G are bounded and

$$\begin{cases} |B(t,x,y,z,\mu)| \le L\left[1 + |x| + |y| + |z| + \left(\int_{\mathbb{R}^d\times\mathbb{R}^p} |(x',y')|^2 d\mu(x',y')\right)^{1/2}\right], \\ |F(t,x,y,z,\mu)| \le L\left[1 + |y| + \left(\int_{\mathbb{R}^d\times\mathbb{R}^p} |y'|^2 d\mu(x',y')\right)^{1/2}\right]. \end{cases}$$

(A3). Σ is uniformly elliptic

$$\Sigma(t,x,y,\mu)\Sigma(t,x,y,\mu)^\dagger \ge L^{-1} I_d$$

and $[0,T] \ni t \hookrightarrow \Sigma(t, 0, 0, \delta_{(0,0)})$ is also assumed to be continuous.

Under (A1–3), there exists a solution $(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \in \mathbb{S}^{2,d} \times \mathbb{S}^{2,p} \times \mathbb{H}^{2,p\times m}$

More Generally

- **Lipschitz coefficients**: existence and uniqueness in small time
- **Lipschitz + Bounded coefficients + Non-degenerate** Σ:
  - existence by Schauder type argument (previous slide)
  - Nice but, **as such**, does not apply to Linear Quadratic Models !
- If FBSDE comes from **MFG model** with
  - **linear dynamics**
  - **convex costs**

  existence + uniqueness

Solutions of Specific Applications

The following applications need special attention:

- **Price Impact Model**:
  - interaction through the controls (**extended** MFG)
- **Congestion + Exit of a Room**:
  - McKean-Vlasov FBSDEs in a bounded domain with boundary conditions
- **C-S Flocking**:
  - non-convex cost function
  - degenerate volatility
  - still, find "*explicit*" decoupling field for μ fixed
- **Krusell-Smith Growth Model**:
  - degenerate diffusion
  - singular coefficients (blow up at origin)

Optimization Problem

Simultaneous Minimization of

$$J^i(\boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N) = \mathbb{E}\left\{\int_0^T f(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i)dt + g(X_T, \overline{\mu}_T^N)\right\}, \quad i = 1, \cdots, N$$

under **constraints** of the form

$$dX_t^i = b(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i)dt + \sigma(t, X_t)dW_t^i + \sigma^0(t, X_t^i) \circ dW_t^0, \quad i = 1, \cdots, N.$$

where:

$$\overline{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$$

GOAL: search for **equilibria**

especially when N is large

Example of Model Requirements

- Each player **cannot** on its own, influence **significantly** the global output of the game
- **Large** number of **statistically identical** players ($N \to \infty$)
- **Closed loop** controls in **feedback** form

$$\alpha_t^i = \phi^i(t, (X_t^1, \cdots, X_t^N)), \qquad i = 1, \cdots, N.$$

- **Restricted** controls in **feedback** form

$$\alpha_t^i = \phi^i(t, (X_t^i, \overline{\mu}_t^N)), \qquad i = 1, \cdots, N.$$

- By symmetry, **Distributed** controls

$$\alpha_t^i = \phi^i(t, X_t^i), \qquad i = 1, \cdots, N.$$

- **Identical** feedback functions

$$\phi^1(t, \cdot\,) = \cdots = \phi^N(t, \cdot\,) = \phi(t, \cdot\,), \qquad 0 \le t \le T.$$

Touted Solution (Wishful Thinking)

- **Identify** a (distributed closed loop) **strategy** ϕ from **effective equations** (from stochastic optimization for large populations)
- Each player is assigned the same function ϕ
- So at each time t, player i take action $\alpha_t^i = \phi(t, X_t^i)$

What is the resulting **population behavior**?

- Did we reach some form of equilibrium?
- If yes, what kind of equilibrium?

Mean Field Game (MFG) Standard Strategy

for the search of **Nash equilibria**

- By **symmetry**, interactions depend upon

  empirical distributions
- When constructing the **best response map**

  ALL stochastic optimizations should be "**the same**"
- When N is **large**
  - empirical distributions should converge
  - capture interactions with limits of empirical distributions
  - **ONE** standard stochastic control problem **for each possible limit**
- Still need a **fixed point**

2006 **Lasry - Lions** (MFG) **Caines - Malhamé - Huang** (NCE)

Conditional Law of Large Numbers

- Search for **effective equations** in the **asymptotic regime** $N \to \infty$
- Then, solve (in this asymptotic regime) for
  - a **Nash equilibrium**?
  - a **stochastic control** problem?
- If we consider **exchangeable equilibria**,$(\alpha_t^1, \cdots, \alpha_t^N)$, then
  - By **LLN**
$$\lim_{N\to\infty} \overline{\mu}_t^N = \mathbb{P}_{X_t^1 | \mathcal{F}_t^0}$$
  - **Dynamics** of player 1 (or any other player) becomes
$$dX_t^1 = b(t, X_t^1, \mu_t, \alpha_t^1)dt + \sigma(t, X_t^1)dW_t + \sigma^0(t, X_t) \circ dW_t^0$$
    with $\mu_t = \mathbb{P}_{X_t^1 | \mathcal{F}_t^0}$.
  - **Cost** to player 1 (or any other player) becomes
$$\mathbb{E}\left\{\int_0^T f(t, X_t, \mu_t, \alpha_t^1)dt + g(X_T, \mu_T)\right\}$$

MFG Problem with Common Noise

1. **Fix** a measure valued $(\mathcal{F}_t^0)$-adapted process (μ_t) in $\mathcal{P}(\mathbb{R})$;
2. Solve the standard **stochastic control problem**

$$\hat{\alpha} = \arg\inf_{\alpha} \mathbb{E}\left\{\int_0^T f(t, X_t, \mu_t, \alpha_t)dt + g(X_T, \mu_T)\right\}$$

   subject to

$$dX_t = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t)dW_t + \sigma^0(t, X_t) \circ dW_t^0;$$

3. **Fixed Point Problem**: determine (μ_t) so that

$$\forall t \in [0, T], \quad \mathbb{P}_{X_t | \mathcal{F}_t^0} = \mu_t \quad a.s.$$

Once this is done, **if** $\hat{\alpha}_t = \phi(t, X_t)$, go back to N player game and show that:

$$\alpha_t^{j*} = \phi^*(t, X_t^j), \qquad j = 1, \cdots, N$$

form an **approximate Nash equilibrium** for the game with N players.

Recent Results by Probabilistic Methods

R.C. - F. Delarue (two-volume book to appear)

- MFG version of Cucker-Smale flocking model
- Crowd motion with congestion, e.g. exit of a room
- Price impact model
- Diffusion form of Krusell - Smith growth model
- Interacting OU model for systemic risk with delay (**RC - Fouque**)
- MFGs of timing for bank runs (**R. - Lacker**)
- MFGs with Major and Minor players (**RC - Zhu**), in finite spaces (**R.C. - Wang**)

In each case, we **prove existence**, sometimes **uniqueness**, often give **numerical illustrations**, unfortunately (so far) computations rarely **stable** away from LQ models.

Solving MFGs by Solving FBSDEs of McKean-Vlasov Type

Probabilistic Approach: First Prong

BSDE Representation of the Value Function: $Y_t = V^\mu(t, X_t)$
(Reduced) Hamiltonian

$$H^{\mu}(t,x,y,\alpha) = b(t,x,\mu_t,\alpha)\cdot y + f(t,x,\mu_t,\alpha)$$

Determine (assume existence of):

$$\hat\alpha^{\mu}(t,x,y) = \arg\inf_{\alpha} H^{\mu}(t,x,y,\alpha)$$

Inject in **FORWARD** and **BACKWARD** dynamics and **SOLVE**

$$\begin{cases} dX_t = b(t,X_t,\mu_t,\hat\alpha^{\mu}(t,X_t,Z_t\sigma^{-1}(t,X_t)))dt + \sigma(t,X_t)dW_t \\ dY_t = -f(t,X,Z_t\sigma^{-1}(t,X_t),\hat\alpha^{\mu}(t,X_t,Z_t\sigma^{-1}(t,X_t)))dt + Z_t dW_t \end{cases}$$

with $X_0 = x_0$ and $Y_T = g(X_T, \mu_T)$

Standard **FBSDE** (for each **fixed** $t \hookrightarrow \mu_t$)

Probabilistic Approach: Second Prong

Stochastic Maximum Principle: $Y_t = \partial_x V^\mu(t, X_t)$

Inject in **FORWARD** and **BACKWARD** dynamics and **SOLVE**

$$\begin{cases} dX_t = b(t, X_t, \mu_t, \hat{\alpha}^{\boldsymbol{\mu}}(t, X_t, Y_t))dt + \sigma(t, X_t)dW_t \\ dY_t = -\partial_x H^{\boldsymbol{\mu}}(t, X_t^\alpha, Y_t, \hat{\alpha}^{\boldsymbol{\mu}}(t, X_t, Y_t))dt + Z_t dW_t \end{cases}$$

with $X_0 = x_0$ and $Y_T = \partial_x g(X_T, \mu_T)$

Standard **FBSDE** (for each **fixed** $t \hookrightarrow \mu_t$)

Fixed Point Step

Solve the **fixed point problem**

$$\boldsymbol{\mu} = (\mu_t)_{0\le t\le T} \quad \longrightarrow \quad \mathbf{X}^{\boldsymbol{\mu}} = (X_t)_{0\le t\le T} \quad \longrightarrow \quad \boldsymbol{\nu} = (\nu_t = \mathbb{P}_{X_t})_{0\le t\le T}$$

Note: if we enforce $\mu_t = \mathbb{P}_{X_t}$ for all $0 \le t \le T$ in FBSDE we have

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}^{\mathbb{P}_{X_t}}(t, X_t, Y_t))dt + \sigma dW_t \\ dY_t = -\Psi(t, X, Y_t, \mathbb{P}_{X_t})dt + Z_t dW_t \end{cases}$$

with

$$X_0 = x_0 \qquad \text{and} \qquad Y_T = G(X_T, \mathbb{P}_{X_T})$$

FBSDE of (Conditional) McKean-Vlasov type !!!

Very difficult

FBSDEs of McKean-Vlasov Type

RC - Delarue

$$\begin{cases} dX_t = B(t, X_t, Y_t, Z_t, \mathbb{P}_{(X_t, Y_t)})dt + \Sigma(t, X_t, Y_t, \mathbb{P}_{(X_t, Y_t)})dW_t \\ dY_t = -\Psi(t, X, Y_t, Z_t, \mathbb{P}_{(X_t, Y_t)})dt + Z_t dW_t \end{cases}$$

- **Lipschitz coefficients**: existence and uniqueness in small time
- **Lipschitz + Bounded coefficients + Non-degenerate** Σ: existence
- FBSDE from **linear MFG** with **convex costs**: existence + uniqueness

Solutions of Specific Applications

- **Price Impact Model**:
  - interaction through the controls (**extended** MFG)
- **Congestion + Exit of a Room**:
  - McKean-Vlasov FBSDEs in a bounded domain with boundary conditions
- **C-S Flocking**:
  - non-convex cost function
  - degenerate volatility
  - still, find "*explicit*" decoupling field for μ fixed
- **Krusell-Smith Growth Model**:
  - degenerate diffusion
  - singular coefficients (blow up at origin)

Back to the MFG Problem

- For $\boldsymbol{\mu} = (\mu_t)_t$ **fixed**, assume decoupling field $u^{\boldsymbol{\mu}} : [0, T] \times \mathbb{R}^d \hookrightarrow \mathbb{R}$ exists so that
$$Y_t = u^{\mu}(t, X_t)$$
- **Dynamics** of X
$$dX_t = b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, u^{\mu}(t, X_t)))dt + \sigma dW_t$$
- **In equilibrium** $\mu_t = \mathbb{P}_{X_t}$
$$Y_t = u^{\mathbb{P}_{X_t}}(t, X_t).$$
- **Could the function**
$$(t, x, \mu) \hookrightarrow U(t, x, \mu) = u^{\mu}(t, X_t)$$
be the solution of a PDE, with time evolving in **one single direction**?

MASTER EQUATION touted by **P.L. Lions** in his lectures. (Lecture II).