Walking within growing domains: recurrence versus transience

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[1] Jointly with Ruojun Huang, Ben Morris, Yuval Peres, Vladas Sidoravicius.

Conductance models (reversible Markov chains)

$\mathbb{G} = (V, E)$ locally finite, connected graph.
Edge conductances $\{\pi(x, y) > 0 : (x, y) \in E\}$.

Irreducible Markov chain $(X_t, t \in \mathbb{N})$ of transition probabilities:

$$p(t, x; t+1, y) = \frac{\pi(x, y)}{\pi(x)}, \qquad \forall (x, y) \in E, \qquad t \geq 0\,.$$

Reversing measure $\pi(x) = \sum_y \pi(x, y)$.

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$$\text{Occupation measure:} \qquad N_y = \sum_{t=1}^{\infty} \mathbf{1}_y(X_t)$$

Recurrence/Transience

$$\forall x, y \qquad \mathbb{P}_x(N_y = \infty) = 1 \Leftrightarrow \mathbb{E}_x(N_y) = \infty \Leftrightarrow \exists y \quad \mathbb{P}_y(N_y \geq 1) = 1\,.$$

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$$\mathbb{P}(X_{t+1} = y | X_t = x) := p^{(t)}(x, y) = \frac{\pi^{(t)}(x, y)}{\pi^{(t)}(x)} \,.$$

$\{\pi^{(t)}(x, y) > 0 : (x, y) \in E\}$, independent of $\{X_s, s \geq 0\}$.
$\pi^{(t)}$-recurrence: $q_{xy} := \mathbb{P}_x(N_y = \infty) = 1, \quad \forall x, y$.

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Rich behavior [ABGK '08]: $\mathbb{G} = \mathbb{N}$, $\exists\, \underline{\pi} \leq \pi^{(t)} \leq \overline{\pi}$ such that:

- $\underline{\pi} \,\&\, \overline{\pi}$ recurrent, $\pi^{(t)} \downarrow$ having $q_{yy} \in (0,1)$ (no 0-1 law, [Ex. 4.5]).
- $\underline{\pi} \,\&\, \overline{\pi}$ recurrent, $\pi^{(t)} \downarrow$ is transient [Ex. 4.6]
- $\underline{\pi} \,\&\, \overline{\pi}$ transient, $t \mapsto \pi^{(t)}$ non-monotone & recurrent [Ex. 3.6]

When $\mathbb{G} = \mathbb{T}$ ([ABGK '08]):

- $\pi^{(t)} \uparrow \overline{\pi}$ recurrent $\Rightarrow$ $\pi^{(t)}$-recurrence [Thm. 5.1] $(1_\star)$
- $\pi^{(t)} \downarrow \underline{\pi}$ transient $\Rightarrow$ $\pi^{(t)}$-transience [Thm. 5.2] $(2_\star)$
- $\pi^{(t)} \uparrow$, $\underline{\pi} \,\&\, \overline{\pi}$ transient $\Rightarrow$ $\pi^{(t)}$-transience [Thm. 4.2, $\mathbb{N}$] $(3_\star)$
- $\pi^{(t)} \downarrow$, $\underline{\pi} = \epsilon\overline{\pi}$ recurrent $\Rightarrow$ $\pi^{(t)}$-recurrence [Thm. 4.4, $\mathbb{N}$] $(4_\star)$

Proof: Unit flows yield potential $F_t(v)$, i.e. $\pi^{(t)}$-harmonic on $\mathbb{T}\backslash\{o\}$ with $t \mapsto F_t(v)$ monotone. Thereby, use optional stopping for $F_t(X_t)$ sub/supMG.

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Special case (Open): $\pi^{(t)} \in [\epsilon, 1]$, $\mathbb{G} = \mathbb{Z}^d$ ([DHMP '15] proved $(3_\star)$).

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$(1_\star)$-$(4_\star)$ hold even if $\pi^{(t)}(\cdot,\cdot)$ adapted to $\{X_s, s \le t\}$
BUT $(1_\star)$ fails for $\mathbb{G} = \mathbb{Z}^2$ and some adaptive $\pi^{(t)}$ [Sec. 6, ABGK].

- DSRW periodic chain (recall $p(0,x;2t+1,x)=0$, $t\in\mathbb{N}$);
  γ-lazy: $\pi^{(t)}(x,x)\geq\gamma$, $\forall x,t$, is a-periodic.

- $\{X_s, s\geq 0\}$ CSRW, jumps at T_k w.p. $p^{(T_k)}(X_{T_k^-},y)$ for
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- $\{X_s, s\geq 0\}$ VSRW, jumps at T_k w.p. $p^{(T_k)}(X_{T_k^-},y)$ for
  independent $(T_{k+1}-T_k)$ of the Exp($\pi^{(t)}(X_{T_k})$) density at t.
  VSRW has constant (in t,x), reversing measure.

- Time-invariant model: VSRW/CSRW time-changes of same DSRW
  Time-varying model: possibly recurrent VSRW, transient CSRW or vice verse!

Gaussian heat kernel estimates

Special case: $\mathbb{G} = \mathbb{Z}^d$, $\pi^{(t)} \in [\epsilon, 1]$.

GHKE: $\exists c_j \in (0, \infty)$ such that $\forall t \geq |x-y| \vee 1$,

$$c_1 t^{-d/2} e^{-c_2 \frac{|x-y|^2}{t}} \leq p(0, x; t, y) \leq c_3 t^{-d/2} e^{-c_4 \frac{|x-y|^2}{t}} .$$

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Hold for uniformly elliptic parabolic PDE in divergence form
[Aronson '67, after De Giorgi, Nash, Moser '50-'60];
for Laplace-Beltrami operator, equivalent to VD+PI_2 via parabolic Harnack
[Grigor'yan, Saloff-Coste '92]; for Dirichlet forms on metric spaces [Sturm '95];
for γ-lazy $\mathbb{Z}^d$-conductance models with $\pi \in [\epsilon, 1]$ [Delmotte '99];
useful for random walk in random conductances [Biskup '11, Kumagai '14];

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Diagonal ($x = y$) GHKE (+Borel-Cantelli) $\Rightarrow$ recurrence iff $d \leq 2$.

- GHKE holds for time-varying VSRW [Delmotte-Deuschel '05].
- GHKE fails for some time-varying CSRW and γ-lazy DSRW: ballistic on $\mathbb{Z}$, recurrent on $\mathbb{Z}^2 \times \mathbb{N}$ [Huang-Kumagai '15]. (non-monotone $t \mapsto \pi^{(t)}(x)$, does not contradict [Conj. 7.1, ABGK]).

Evolving sets: $t \mapsto \pi^{(t)}(x)$ non-decreasing [DHMP '15]

Admissible sites $V_t := \{y \in V : \pi^{(t)}(y) > 0\}$, non-decreasing in t.

$(U_t, t \in \mathbb{N})$, i.i.d. $U(0,1)$, independent of $\{X_s, s \geq 0\}$.

Evolving set process $\{S_t, t \in \mathbb{N}\}$: $S_0 = \{x\}$, $x \in V_0$,

$$S_{t+1} = \{y \in V_{t+1} : \frac{\pi^{(t)}(S_t, y)}{\pi^{(t+1)}(y)} \geq U_{t+1}\}.$$

Time invariant case: [Morris-Peres '05], applicable for M.C. mixing time (also sized-biased version [Diaconis-Fill '90] for strong stationary times).

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- $\pi^{(t)}(S_t)$ is a martingale, $\forall x, y, t \geq 0$

$$p(0, x; t, y) = \frac{\pi^{(t)}(y)}{\pi^{(0)}(x)} \mathbb{P}_{\{x\}}(y \in S_t).$$

γ-lazy DSRW or CSRW; $\pi^{(t)}(x) \uparrow$ uniformly bounded.

Isomerimetric growth $(d > 1)$:

$$\kappa_u := \inf_{A \subset V_u, 0<|A|<\infty} \left\{ \frac{\pi^{(u)}(A, A^c)}{\pi^{(u)}(A)^{(d-1)/d}} \right\}.$$

Example: $\mathbb{G} = \mathbb{Z}^d$, $\pi^{(t)} \in [\epsilon, 1] \quad \Longrightarrow \quad \inf_u\{\kappa_u\} \geq \delta_d(\epsilon) > 0.$

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Evolving sets $\Rightarrow$ diagonal GHK upper-bound [DHMP '15]:

$$\pi^{(0)}(x)p(0, x; t, y) \leq c_3 \Big(\sum_{u<t} \kappa_u^2 \Big)^{-d/2} \qquad \forall x \in V_0, y \in V_t, \quad t \geq 1.$$

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Consequences $(d > 2)$:

- $\mathbb{G} = \mathbb{Z}^d$, any $\pi^{(t)} \in [\epsilon, 1]$, $\pi^{(t)} \uparrow$ is transient ([ABGK $(3_\star)$]).
- $\mathbb{D}_0 = \mathcal{C}_p$ the ∞-cluster of bond percolation at $p > p_c(\mathbb{Z}^d)$. Any γ-lazy DSRW on edge-set $\mathbb{D}_t \uparrow$ is transient.

[DHS '14] study DSRW on connected $\mathbb{D}_t \uparrow \mathbb{Z}^d$, such that:

$$f(t)\mathbb{B}_1 \cap \mathbb{Z}^d \subseteq \mathbb{D}_t \subseteq f(t)\mathbb{B}_c \cap \mathbb{Z}^d \,,$$

some c finite and scale $0 < f(t) \uparrow \infty$.

[Conj. 1.2, DHS]: DSRW on $\mathbb{D}_t$ is recurrent $\Leftrightarrow J_f := \int \frac{dt}{f(t)^d} = \infty$.
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[Thm. 1.4, DHS]: $J_f < \infty \Rightarrow$ DSRW transient.
$J_f = \infty \Rightarrow$ DSRW recurrent in case $\mathbb{D}_t = f(t)\mathbb{K} \cap \mathbb{Z}^d$
($\mathbb{K} \subset \mathbb{R}^d$ assumed $\star$-shaped, bounded uniform domain,
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($\mathbb{K} \subset \mathbb{R}^d$ assumed $\star$-shaped, bounded uniform domain,
$f(\cdot)$ assumed piece-wise constant, well separated scales).

Proof: By invariance principle reduce to recurrence of reflected Brownian motion on growing domains $\mathbb{K}_t = f(t)\mathbb{K}$ (see [BC '11,BCS '04]);
Solve for $\mathbb{K} = \mathbb{B}_1$ (radial symmetry);
Extend to $\mathbb{K}$ by Neumann heat kernel comparisons (see [Pascu '11]).

Universality for $\textsc{dsrw}$ on growing domains

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$\mathbb{G}_t \uparrow$, $\mathbb{G}'_t \uparrow$, with $\mathbb{G}'_\infty$ of uniformly bounded degrees.
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$\exists$ $\mathbb{G}'_t$ of unbounded degrees, for which [Conj. 1.8, DHS] fails.

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[DHS '14b]: DSRW $\{X_t\}$ on $\mathbb{G}_t \uparrow \mathbb{G}_\infty \subseteq \overline{\mathbb{G}}$ of bounded degrees

$$(\star) \quad \mathbb{B}_{\overline{\mathbb{G}}}(X_t, 1) \subseteq \mathbb{G}_t \quad \Longrightarrow \quad \mathbb{G}_{t+1} = \mathbb{G}_t$$

- Open By Touch (OBT): $\mathbb{G}_{t+1} = \mathbb{G}_t \cup \mathbb{B}_{\overline{\mathbb{G}}}(X_t, 1)$
- Partial Open By Touch (POBT): $\mathbb{G}_{t+1} \subseteq \mathbb{G}_t \cup \mathbb{B}_{\overline{\mathbb{G}}}(X_t, 1)$
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[Prop. 1.9, DHS-b]: $\overline{\mathbb{G}} = \mathbb{Z}^d$, $d > 2$.
$m_k = |(\mathbb{G}_0)^c \cap \partial \mathbb{B}_{\overline{\mathbb{G}}}(X_0, k)|$, $\quad \sum_k \frac{m_k}{k^{d-2}} < \infty \Rightarrow$ POBT transient.
[Conj. 1.12, DHS-b]: $\mathbb{G}_0 = \mathcal{C}_p$, $p > p_c \Rightarrow$ OBT transient (open).

- Once edge-reinforced walk on $\mathbb{G}$ is a special case of POBT!
- For finite $\mathbb{G}_0$ the specifics of the POBT matter.

Expanding glassy spheres, almost-regular shape

$\overline{\mathbb{D}}_k = \mathbb{B}_{ck} \cap \mathbb{Z}^d$, $d \geq 2$, $\quad X_0 = 0$, $\quad c \geq 1$, $\quad N_k \geq 1$.

EGS: $\mathbb{D}_t = \overline{\mathbb{D}}_k$ for $t \in [\tau_k, \tau_{k+1})$, $\tau_1 = 0$, τ_{k+1} first after N_k-th visit to $\partial\overline{\mathbb{D}}_k$.

[Prop. 1.14, DHS-b]: EGS transient $\Leftrightarrow J := \sum_k N_k k^{1-d} < \infty$;
EGS recurrent $\Leftrightarrow J = \infty$.

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Defn: $\mathbb{D}_t \uparrow \mathbb{D}_\infty \subseteq \mathbb{Z}^d$, $d \geq 2$ admit c-almost-regular shape $\mathbb{K}$, if:

- $f(t)\mathbb{K} \cap \mathbb{Z}^d \subset \mathbb{D}_t$, $1 \leq f(t) \uparrow$
- $\sup_{z \in \mathbb{D}_t} \{d_{\mathbb{D}_t}(z, f(t)\mathbb{K})\} \leq c \log f(t)$

[Prop. 1.18, DHS-b]: $\exists c_d > 0$ s.t. for any $\mathbb{D}_0$ finite
POBT on $\{\mathbb{D}_t\}$ admitting c_d-almost-regular shape $\mathbb{B}$, must be recurrent.

Open: prove c-almost-regular shape for even one non-trivial POBT!

Thank you!