Markov Dynamics on Macdonald Processes

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October 4, 2013

Main goal

Present axiomatic approach to known models of Markov dynamics, and get new examples out of it

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A motivating example from random matrices: appearance of Dyson's Brownian motion (originally, from GUE random matrices [Dyson '62]):

1. Path transformation of independent Brownian motions related to Robinson–Schensted–Knuth correspondence [O'Connell '03]
2. Warren's construction '07

Both extend to *hierarchies* of diffusions which are *different*, but have the *same* fixed-time distributions (corresponding to GUE corners). Why? Are there any other diffusions with these properties?

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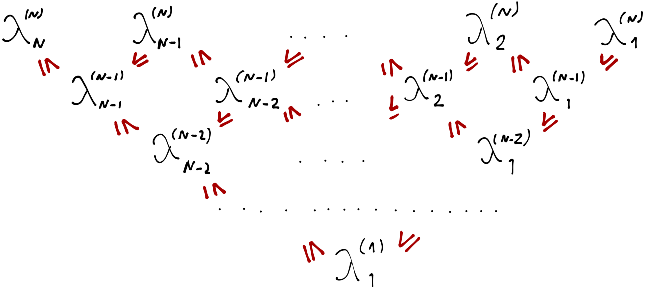
I will explain a discrete version of the problem, and a solution.

Outline

1. "Schur level"
   - Push-block dynamics ("Warren")
   - RSK dynamics ("Path transformation")
   - **Unifying axioms**
   - **New RSK correspondences**
2. "Macdonald level"
   - From Schur to Macdonald
   - q-deformed 1d particle systems: **new examples**
   - **Randomized insertion algorithm for triangular matrices over a finite field**

Interlacing integer arrays

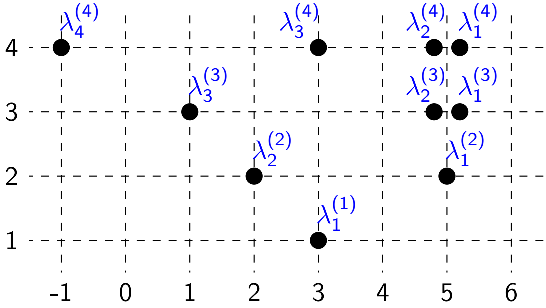
Interlacing integer arrays (= Gelfand-Tsetlin schemes)



Main object: continuous-time Markov dynamics on the space of interlacing integer arrays.

Interlacing integer arrays $\longleftrightarrow$ particles in 2 dimensions

$$\begin{array}{ccccccc} -1 & & 3 & & 5 & & 5 \\ & 1 & & 5 & & 5 & \\ & & 2 & & 5 & & \\ & & & 3 & & & \end{array}$$



1 particle at level 1,
2 particles at level 2, etc.

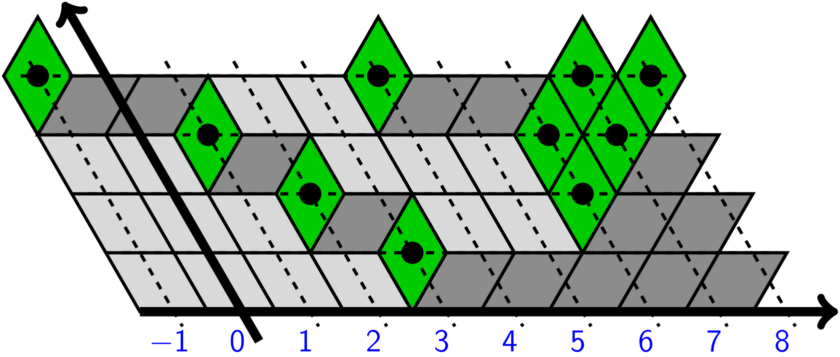
Interlacing integer arrays $\longleftrightarrow$ lozenge tilings

+0 … +(N-2) +(N-1)

$$\begin{array}{ccccccc} -1 & & 3 & & 5 & & 5 \\ & 1 & & 5 & & 5 & \\ & & 2 & & 5 & & \\ & & & 3 & & & \end{array}$$

shift coordinates so that in each row they become distinct

$$\begin{array}{ccccccc} -1 & & 4 & & 7 & & 8 \\ & 1 & & 6 & & 7 & \\ & & 2 & & 6 & & \\ & & & 3 & & & \end{array}$$

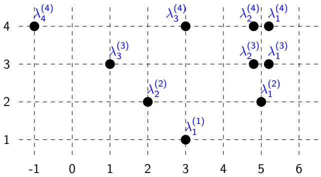


Two examples of dynamics on interlacing arrays:

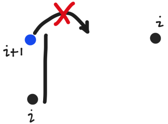
- Push-block dynamics
- Robinson–Schensted–Knuth (RSK) dynamics
- Common properties of the two dynamics

Push-block dynamics [Borodin–Ferrari ’08], [Gorin–Shkolnikov ‘12]

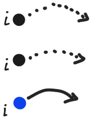
1. Each particle $\lambda_j^{(k)}$ jumps to the right by one according to an independent exponential clock of rate 1.



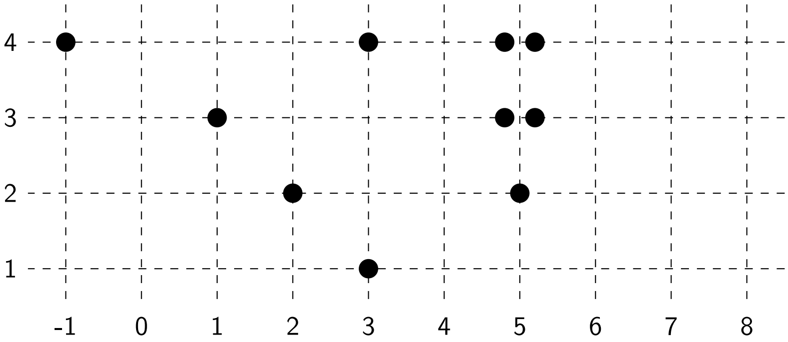
2. If it is blocked from below, there is no jump



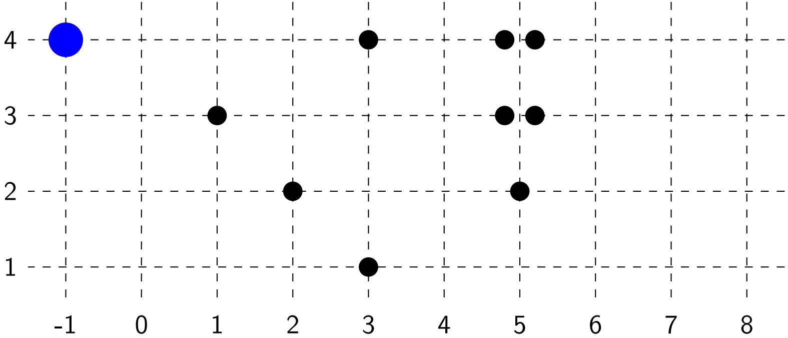
3. If violates interlacing with above, it pushes the above particles



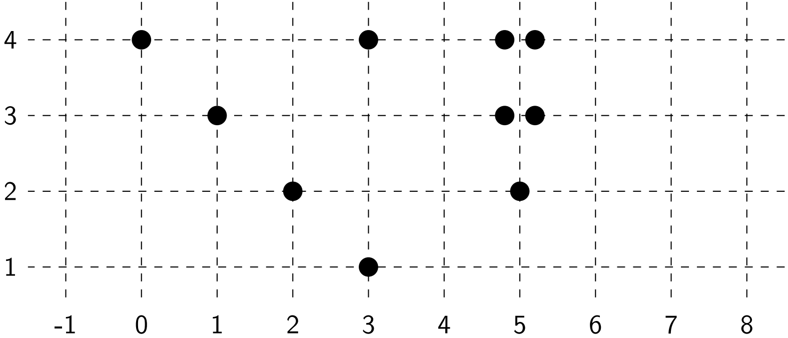
Push-block dynamics



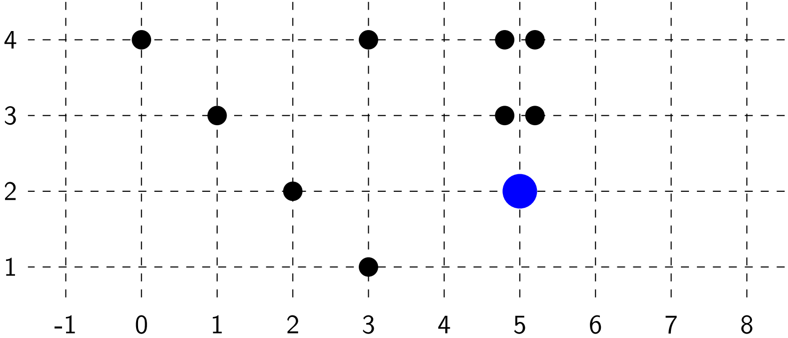
Push-block dynamics



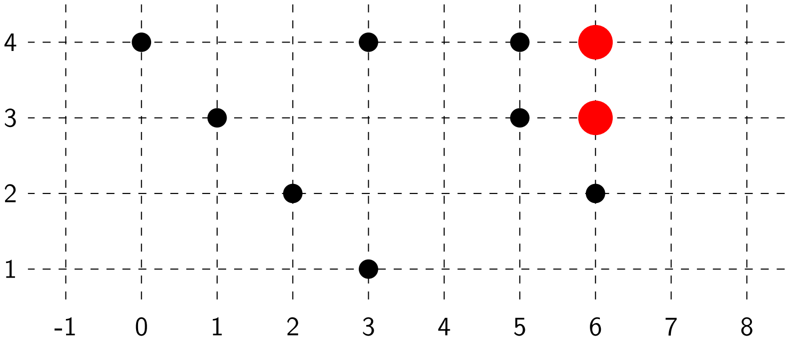
Push-block dynamics



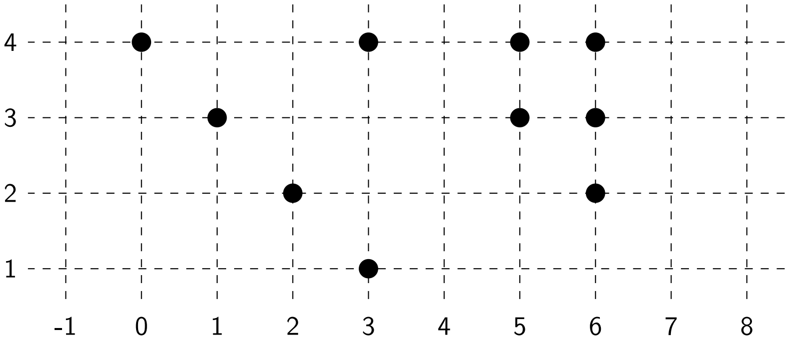
Push-block dynamics



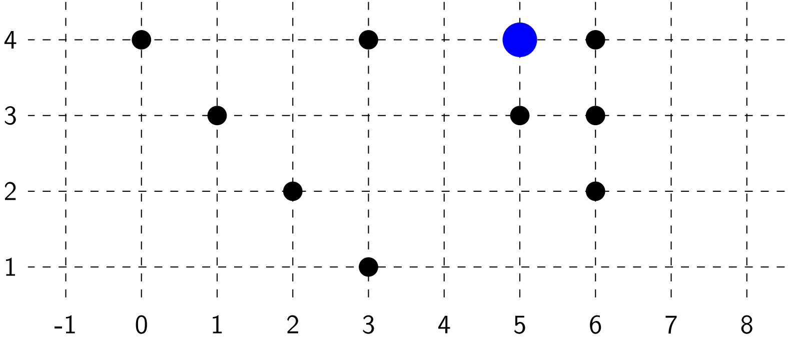
Push-block dynamics



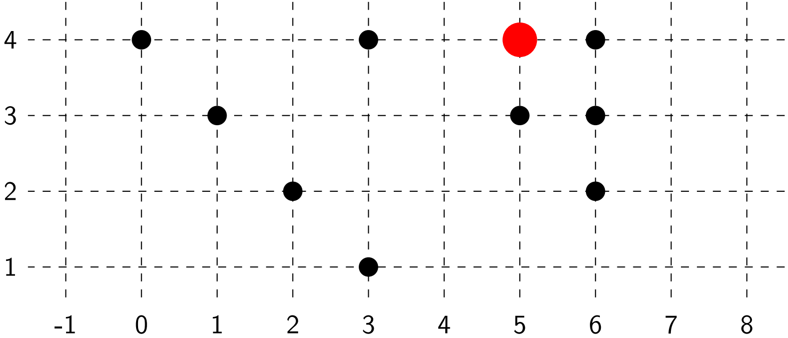
Push-block dynamics



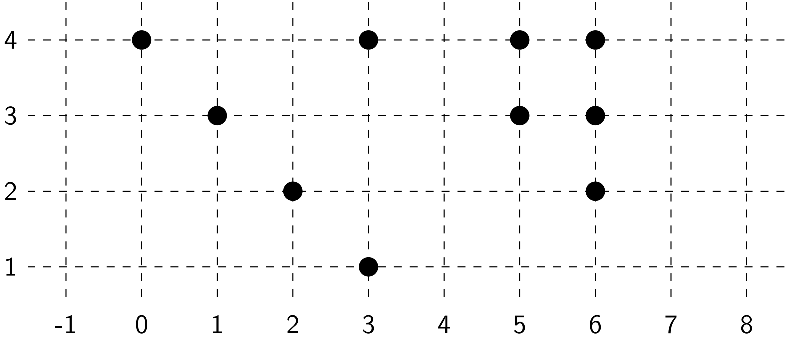
Push-block dynamics



Push-block dynamics

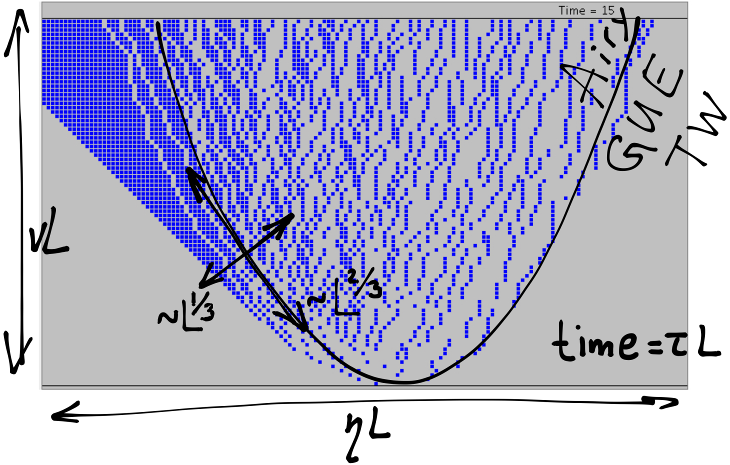


Push-block dynamics



Simulation

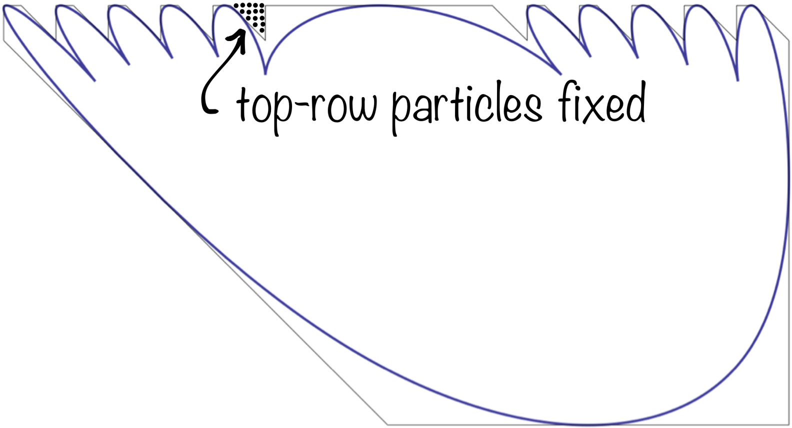
Push-block dynamics: KPZ universality [BF '08]



+ fluctuations $\sim L^{1/3}$ with time
(L — large parameter)

Remark: Other tiling models

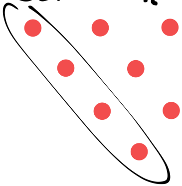
Scaling orders $L^{1/3} - L^{2/3}$, GUE Tracy–Widom distribution and Airy process found in other models of random lozenge tilings: [Okounkov–Reshetikhin '07], [Baik–Kriecherbauer–McLaughlin–Miller '07], **[P. '12]**



1d projections of the push-block dynamics

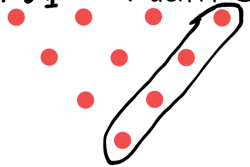
TASEP and PushTASEP

TASEP: $x_n = \lambda_n^{(n)} - n$



Markovian projection to the leftmost particles — TASEP

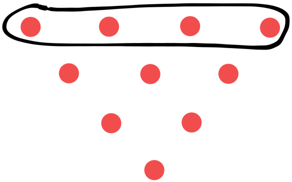
$x_n = \lambda_1^{(n)} + n$: PushTASEP



Markovian projection to the right-most particles — PushTASEP

"Discrete version" of Dyson's Brownian motion

noncolliding Poisson processes



$x_n = \lambda_n^{(N)} - n$

Started from the empty initial state $\lambda_j^{(k)} = 0$, the evolution of the particles in each Nth row is Markovian:

- Rate 1 Poisson processes conditioned never to intersect;
- Equivalently, Doob's h-transform of independent Poisson processes, with $h(x_1, \ldots, x_N) = \prod_{i<j}(x_i - x_j)$.

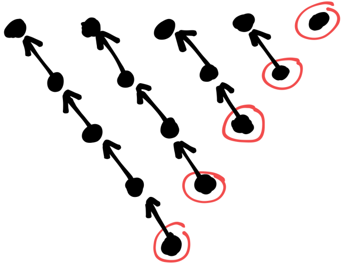
Two examples of dynamics on interlacing arrays:

- Push-block dynamics
- Robinson–Schensted–Knuth (RSK) dynamics
- Common properties of the two dynamics

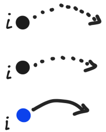
RSK dynamics [Johansson '99,'02], [O'Connell '03]

1. Each rightmost particle $\lambda_1^{(k)}$ jumps to the right by one according to an independent exponential clock of rate 1.

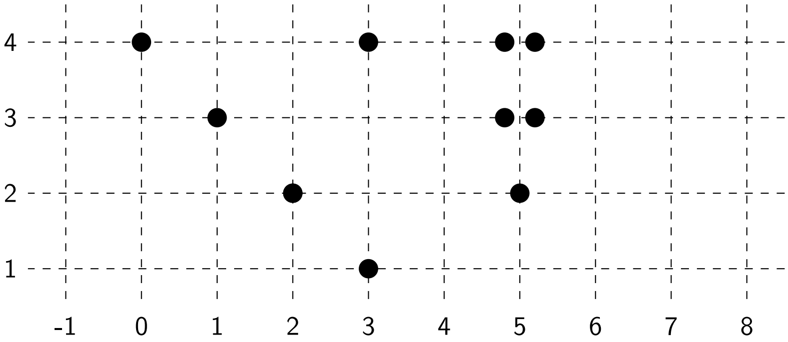
2. When any particle $\lambda_j^{(h)}$ moves, it triggers either the move $\lambda_j^{(h+1)} \mapsto \lambda_j^{(h+1)} + 1$, or $\lambda_{j+1}^{(h+1)} \mapsto \lambda_{j+1}^{(h+1)} + 1$ (exactly one of them).



The second one is chosen generically, while the first one is chosen only if $\lambda_j^{(h+1)} = \lambda_j^{(h)}$, i.e., if the move $\lambda_j^{(h)} \mapsto \lambda_j^{(h)} + 1$ violated the interlacing constraint (push rule).



RSK dynamics



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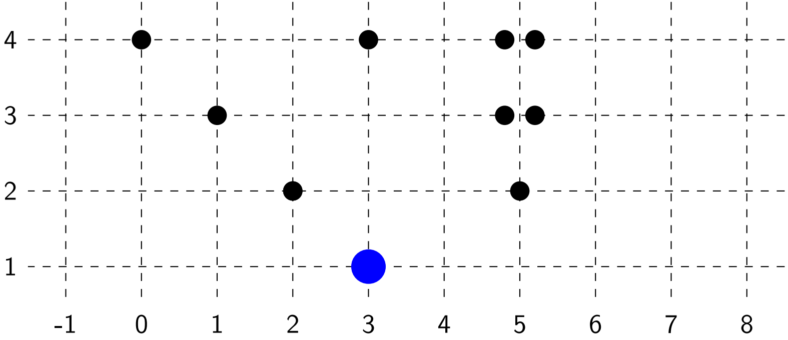

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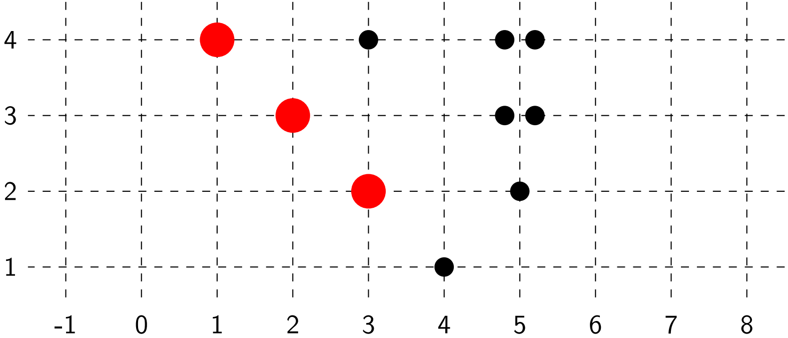
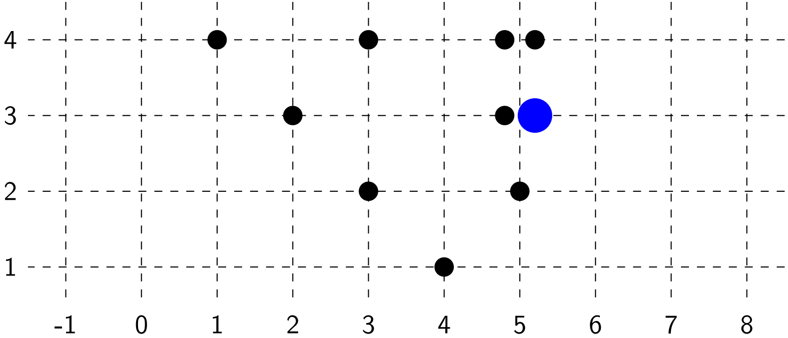


RSK dynamics



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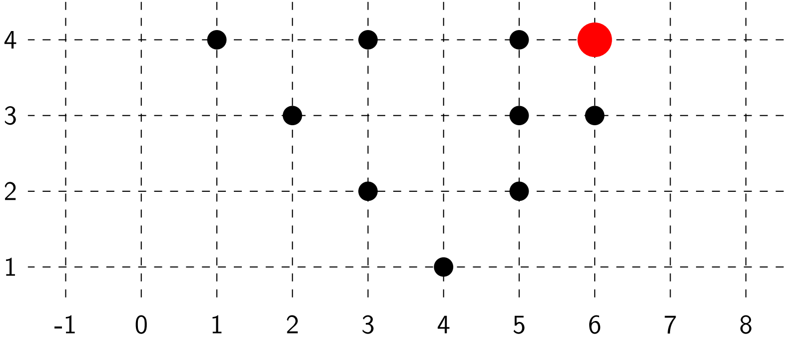
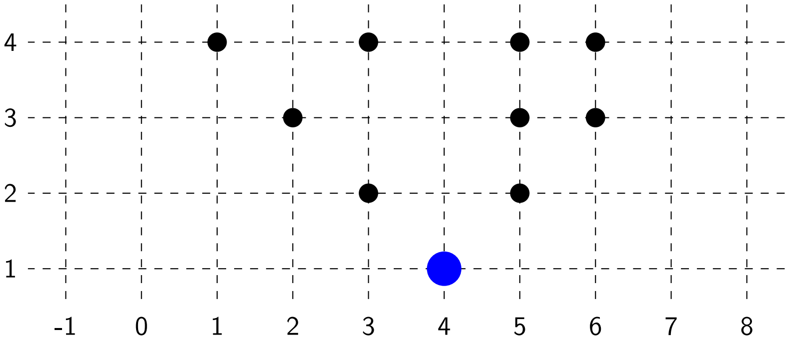


RSK dynamics



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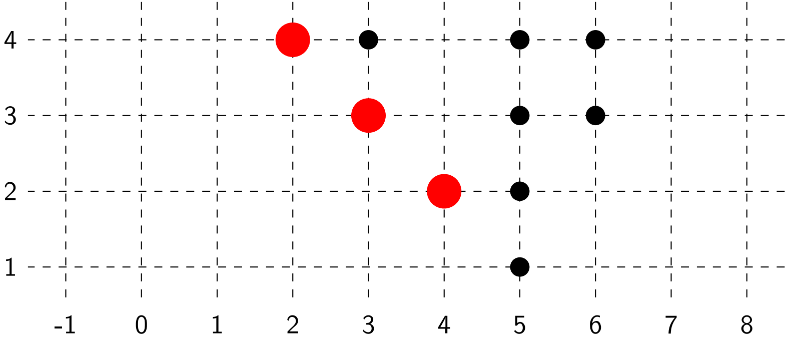
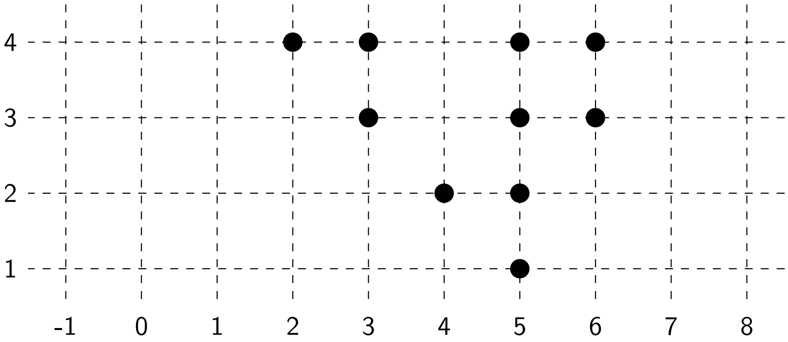
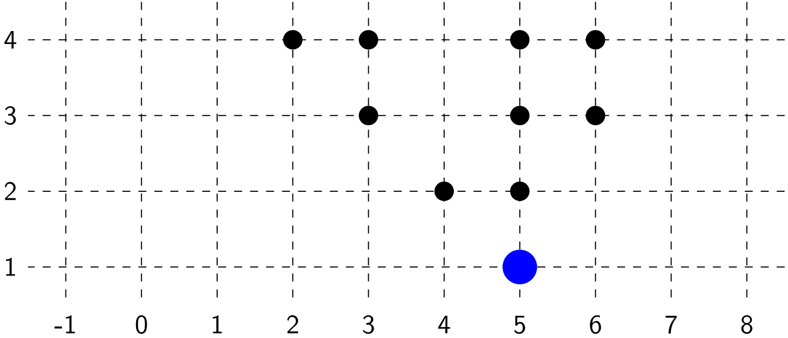


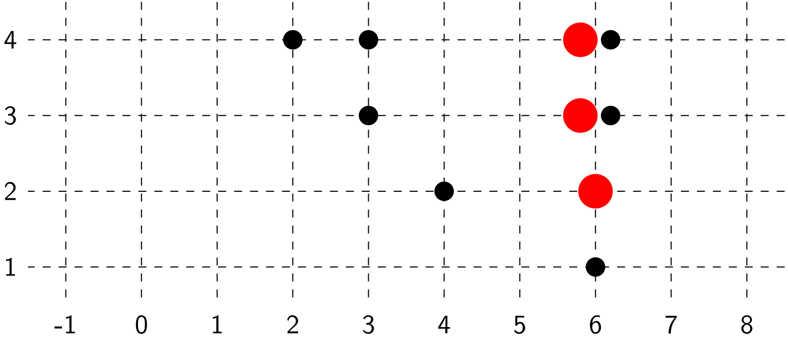
RSK dynamics



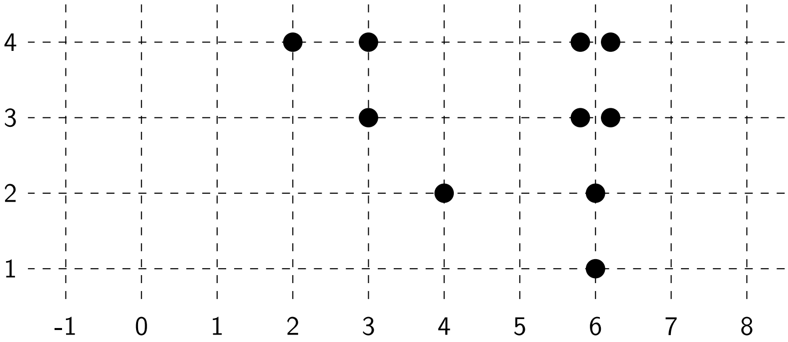
RSK dynamics



RSK dynamics



RSK dynamics



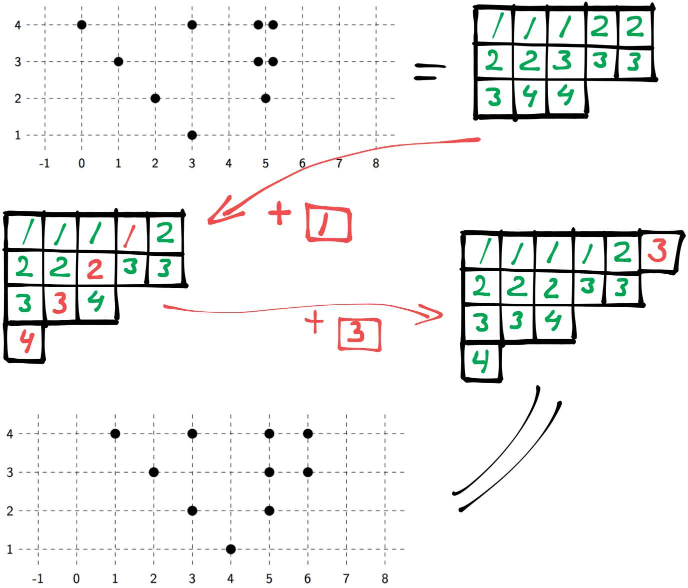
Remark: Classical RSK insertion

RSK is a bijection between words in the alphabet $\{1, 2, \ldots, N\}$ and pairs of Young tableaux (one semistandard and one standard) of the same shape.

Interlacing arrays $\longleftrightarrow$
semistandard Young tableaux ("P-tableaux")

Independent jump at level h $\longleftrightarrow$
RSK-insert letter h into the tableau.

Remark: Classical RSK insertion

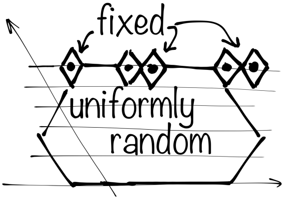


Two examples of dynamics on interlacing arrays:

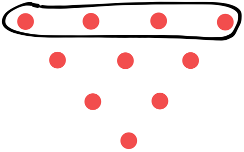
1. Push-block dynamics
2. Robinson–Schensted–Knuth (RSK) dynamics
3. Common properties of the two dynamics

Common properties of both dynamics

1. "Interaction goes up": evolution of $\{\lambda^{(1)}, \dots, \lambda^{(h)}\}$ is independent of $\{\lambda^{(h+1)}, \dots, \lambda^{(N)}\}$.
2. Preserve the class of Gibbs measures



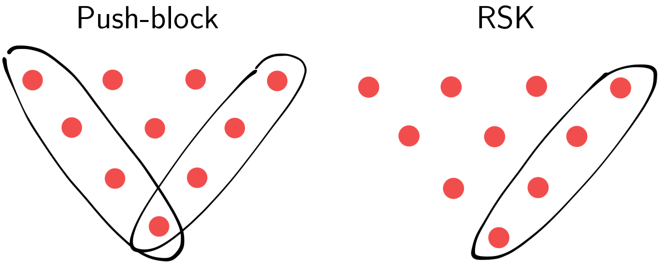
3. Started from a Gibbs measure, each row evolves according to the dynamics of noncolliding Poisson processes ("discrete Dyson's BM")



Common properties of both dynamics

- Started from $\lambda_j^{(k)} = 0$, at time t both dynamics have the same distribution.

- Markovian projections:

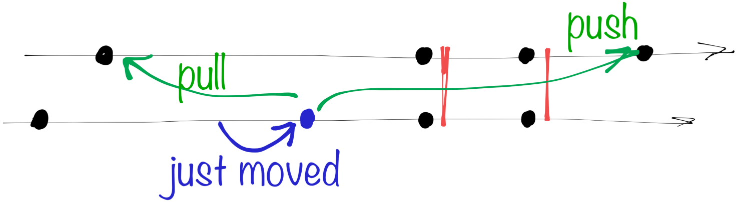


Nearest neighbor dynamics

Nearest neighbor dynamics

We look for other dynamics which satisfy:

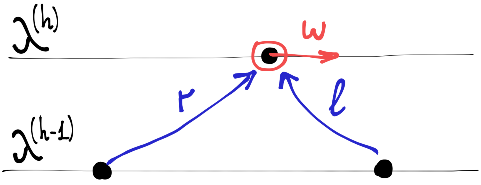
1. "Interaction goes up"
2. Preserve Gibbs measures
3. "discrete Dyson's BM" on floors
4. Nearest neighbor interactions:



(push/pull with some probabilities, do nothing with the complementary probability)

[Borodin-P. '13] — introduce these axioms, and obtain complete classification of nearest neighbor dynamics.

Nearest neighbor dynamics



independent jump rate $w = w(\lambda^{(h-1)}, \lambda^{(h)})$

pushing probabilities (after lower particle jumped) $r = r(\lambda^{(h-1)}, \lambda^{(h)})$ and $\ell = \ell(\lambda^{(h-1)}, \lambda^{(h)})$

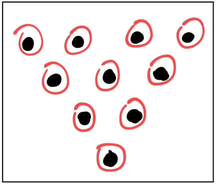
Theorem [Borodin-P. '13]. Nearest neighbor dynamics correspond to solutions of the equations

$$r(\lambda^{(h-1)}, \lambda^{(h)}) + \ell(\lambda^{(h-1)}, \lambda^{(h)}) + w(\lambda^{(h-1)}, \lambda^{(h)}) = 1$$

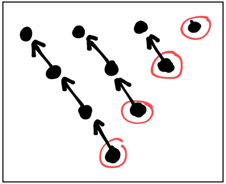
written for all states $\lambda^{(1)}, \ldots, \lambda^{(N)}$ of the array and each particle in it.

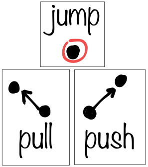
“Basis” dynamics are encoded by pictures such as:

Push-block:

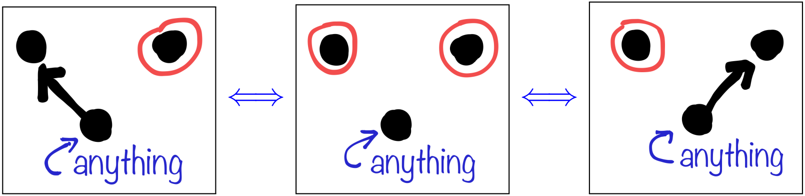


RSK:





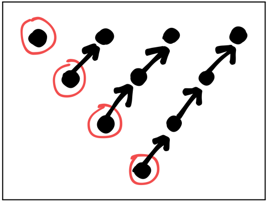
plus local flips



All other dynamics are linear combinations of “basis” ones

“Basis” nearest neighbor dynamics examples

Column (= dual) RSK [O’C ’03]



“Basis” nearest neighbor dynamics examples

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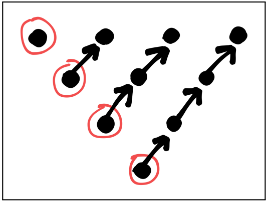
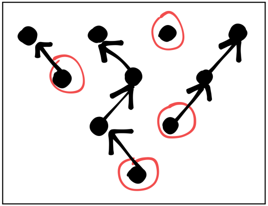


RSK-type ($\Rightarrow$ **we obtain** N! **bijections between words and pairs of tableaux**)



“Basis” nearest neighbor dynamics examples

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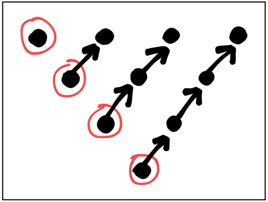
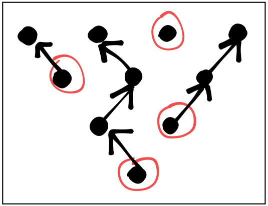
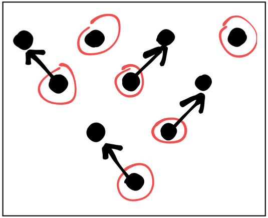


RSK-type (⇒ **we obtain** $N!$ **bijections between words and pairs of tableaux**)



Another example



From Schur to Macdonald

Schur polynomials in dynamics on interlacing arrays

Schur polynomials:

$$s_\mu(x_1,\ldots,x_k)=\frac{\det\left[x_i^{\mu_j+N-j}\right]_{i,j=1}^{k}}{\det\left[x_i^{N-j}\right]_{i,j=1}^{k}},\ \text{where } \mu_1\ge\ldots\ge\mu_k.$$

$s_{\lambda^{(k)}/\lambda^{(k-1)}}$ — skew Schur polynomials.

Distribution of the dynamics — Schur process [Okounkov–Reshetikhin '03]:

$$\mathbb{P}_t\left[\begin{array}{c}\circ\circ\ldots\circ\\ \circ\ldots\circ\\ \vdots\\ \circ\end{array}\rightarrow\begin{array}{c}\lambda^{(N)}_N\ldots\lambda^{(N)}_1\\ \vdots\\ \lambda^{(1)}_1\end{array}\right]=\frac{1}{Z}s_{\lambda^{(1)}}(a_1)\,s_{\lambda^{(2)}/\lambda^{(1)}}(a_2)\ldots s_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)\,s_{\lambda^{(N)}}(\rho_t)$$

$a_1=\ldots=a_K=1$
(a_k = speed at level k)

„Plancherel specialization", t = time

It is a determinantal point process, which is the source of integrability of the model.

Macdonald polynomials

$P_\lambda(x_1,\dots,x_N) \in \mathbb{Q}(q,t)[x_1,\dots,x_N]^{S(N)}$ labeled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$\mathcal{D}_1 = \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i}, \qquad (T_q f)(z) := f(zq),$$

with (generically) pairwise different eigenvalues

$$\mathcal{D}_1 P_\lambda = (q^{\lambda_1} t^{N-1} + q^{\lambda_2} t^{N-2} + \dots + q^{\lambda_N}) P_\lambda.$$

Macdonald polynomials have many remarkable properties (similar to those of Schur polynomials corresponding to $q = t$) including orthogonality, simple reproducing kernel (Cauchy identity), Pieri and branching rules, index/variable duality, etc. There are also simple higher order Macdonald difference operators commuting with $\mathcal{D}_1$.

From Schur to Macdonald

In short, replace all Schur polynomials by Macdonald polynomials. All previous constructions of dynamics work.

Get Markov dynamics on interlacing arrays whose distributions are Macdonald processes [Borodin–Corwin ‘11], [Borodin–Corwin–Gorin–Shakirov ‘13]:

Macdonald polynomials, ordinary and skew

dual basis

$$\mathbb{P}_t\left[\begin{array}{c} \circ\circ\dots\circ \\ \circ\dots\circ \\ \vdots \\ \circ \end{array} \to \begin{array}{c}\lambda^{(N)}_N \dots \lambda^{(N)}_1 \\ \vdots \\ \lambda^{(1)}_1\end{array}\right] = \frac{1}{Z} P_{\lambda^{(1)}}(a_1)\, P_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \dots P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)\, Q_{\lambda^{(N)}}(\rho_t)$$

$a_1 = \dots = a_K = 1$

“Plancherel specialization”, t = time

[Borodin–Corwin ‘11], [O’Connell–Pei ‘12],
[Borodin–P. ‘13] (complete classification of these dynamics)

Symmetric polynomials and related objects

Macdonald polynomials $q, t \in [0,1)$

Eigenfunctions for Ruijsenaars–Macdonald system

Representations of Double Affine Hecke Algebras

q-Whittaker poly's $t=0$

q-deformed quantum Toda lattice

Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$

Hall–Littlewood poly's $q=0$

Spherical functions for p-adic GL(n)

Jack polynomials $t = q^{\beta/2} \to 1$

Eigenfunctions for Calogero–Sutherland

Spherical functions for Riemannian symmetric spaces over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$

Whittaker functions $t=0$, $q \to 1$

Eigenfunctions for quantum Toda lattice

Representations of $GL(n,\mathbb{R})$

Monomial symmetric poly's $q=0$, $t=1$

(simplest symmetric poly's)

Schur polynomials $q=t$

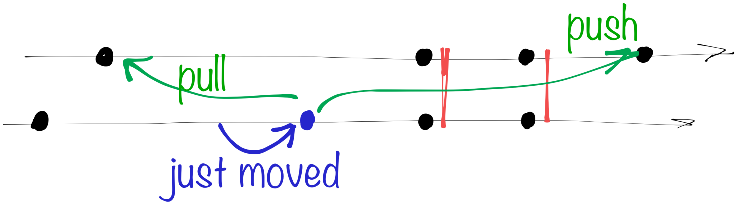
Characters of symmetric and unitary groups

Nearest neighbor dynamics on Macdonald processes:

1. Axioms
2. ($t = 0$) Push-block dynamics and q-TASEP
3. ($t = 0$) q-PushTASEP — a new q-deformed 1d particle system
4. ($q = 0$) Random triangular matrices over a finite field

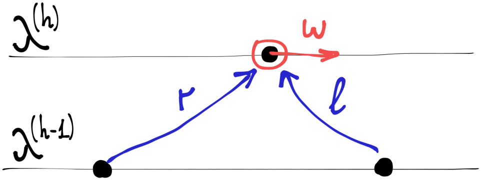
Nearest neighbor dynamics on Macdonald processes

1. "Interaction goes up"
2. Preserve (q,t)-Gibbs measures
3. (q,t)- "discrete Dyson's BM" on floors
4. Nearest neighbor interactions:



(push/pull with some probabilities, do nothing with the complementary probability)

Nearest neighbor dynamics on Macdonald processes



independent jump rate $w = w(\lambda^{(h-1)}, \lambda^{(h)})$

pushing probabilities (after lower particle jumped) $r = r(\lambda^{(h-1)}, \lambda^{(h)})$ and $\ell = \ell(\lambda^{(h-1)}, \lambda^{(h)})$

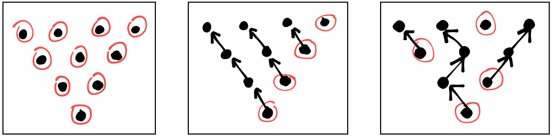
Theorem [Borodin-P. '13]. Nearest neighbor dynamics on Macdonald processes correspond to solutions of the equations

$$T \cdot r(\lambda^{(h-1)}, \lambda^{(h)}) + \tilde{T} \cdot \ell(\lambda^{(h-1)}, \lambda^{(h)}) + w(\lambda^{(h-1)}, \lambda^{(h)}) = S.$$

Here $T, \tilde{T}, S$ are certain coefficients depending on q, t, and also on $\lambda^{(h-1)}, \lambda^{(h)}$.

Nearest neighbor dynamics on Macdonald processes

The "basis" nearest neighbor dynamics are encoded by the same pictures as before.



Not all of the "Schur level" pictures lead to dynamics with nonnegative jump rates. We have to speak about *formal* Markov processes.

Nearest neighbor dynamics on Macdonald processes:

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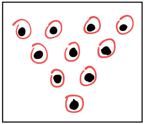
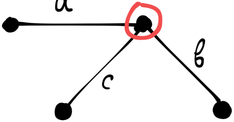
Push-block dynamics [Borodin–Corwin '11]

Let the second Macdonald parameter $t = 0$. The push-block dynamics gives:

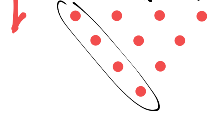


$$\text{rate} = \frac{(1-q^a)(1-q^b)}{1-q^c}$$

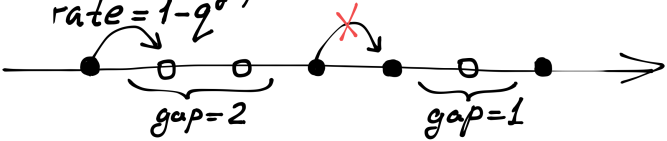


Markovian projection — q-TASEP [BC '11], [BC–Sasamoto '12], [O'Connell–Pei '12], **[BC–P.–Sasamoto '13]**, [Povolotsky '13]

q-TASEP: $x_n = \lambda^{(n)}_n - n$



$$\text{rate} = 1 - q^{\text{gap}}$$



Nearest neighbor dynamics on Macdonald processes:

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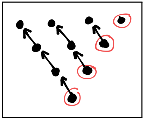
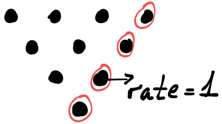
RSK-type dynamics [Borodin–P. ‘13]



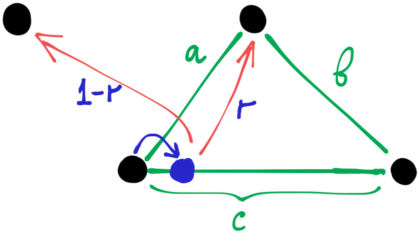
Let the second Macdonald parameter $t = 0$. Then the q-deformation of the classical RSK is:

1. Only the rightmost particles make independent jumps with rate 1



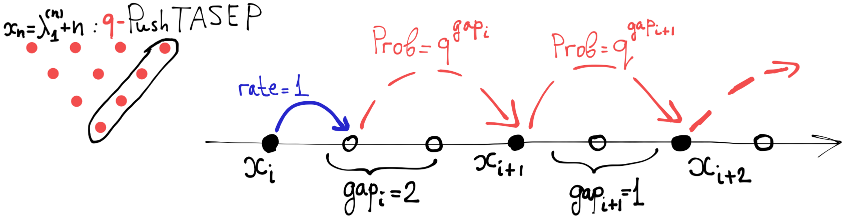
2. If a particle moves, it pushes its immediate upper neighbors with probabilities r and $1 - r$, where

$$r = q^a \frac{1 - q^b}{1 - q^c}$$

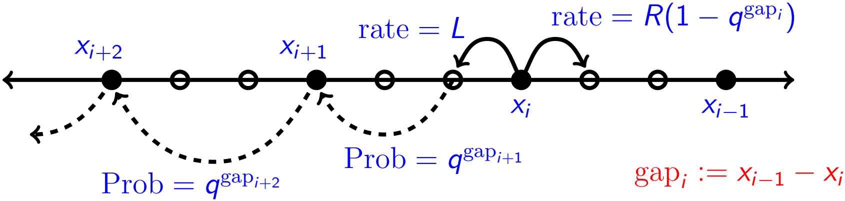


q-PushTASEP [Borodin–P. '13], [Corwin–P. '13]

Another Markovian projection:



Remark: q-PushASEP [Corwin–P. '13]



$R * (q\text{-TASEP, to the right}) + L * (q\text{-PushTASEP, to the left})$

Traffic model (relative to a time frame moving to the right)

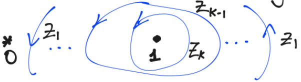
- Right jump = a car *accelerates*. Chance $1 - q^{\text{gap}}$ is lower if another car is in front.
- Left jump = a car *slows down*. The car behind sees the brake lights, and may also quickly slow down, with probability q^{gap} (chance is higher if the car behind is closer).

Remark: q-PushASEP integrability

Theorem [Corwin–P. '13]. q-moment formulas for the q-PushASEP with the step initial condition $x_i(0) = -i$, $i = 1, \ldots, N$.

$$\mathbb{E}\left(\prod_{i=1}^{k} q^{x_{N_i}(t)+N_i}\right) = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint\cdots\oint \prod_{A<B}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k}(1-z_j)^{-N_j}\frac{G_t(qz_j)}{G_t(z_j)}\cdot\frac{dz_j}{z_j}$$

$N_1 \geq N_2 \geq \ldots \geq N_k > 0$



$G_t(z) = e^{t(Rz+Lz^{-1})}$

(obtained via a quantum many body systems approach dating back to H. Bethe '31)

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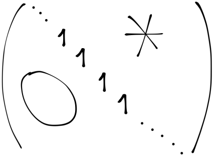
Random triangular matrices over a finite field

Consider the group $\mathbf{U}$ of infinite unipotent upper-triangular matrices over the finite field $F_{t^{-1}}$, where $t \in (0, 1)$, and t^{-1} is a prime power.



[Vershik–Kerov '80s], [Kerov '03]: Problem of classification of probability measures μ on $\mathbf{U}$ which are

- Conjugation-invariant: $\mu(X) = \mu(gXg^{-1})$ for $X \subset \mathbf{U}$ and g a matrix over $F_{t^{-1}}$ which differ from the identity at finitely many positions.
- Ergodic ($=$ extreme as elements of the convex set of all conjugation-invariant measures).

Random triangular matrices over a finite field

Through Jordan normal form of truncations of matrices from **U**, the problem reduces to measures μ_n on Young diagrams $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell$ with fixed number n of boxes. The measures μ_n are related to Hall–Littlewood polynomials (these are Macdonald polynomials with $q = 0$; and t as in $F_{t^{-1}}$).

Conjectural classification of measures μ on **U** [Kerov '03]: measures depend on parameters

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq 0; \qquad \beta_1 \geq \beta_2 \geq \ldots \geq 0; \qquad \sum_{i=1}^{\infty} \left(\alpha_i + \frac{\beta_i}{1-t} \right) \leq 1.$$

These measures $\mu^{\alpha;\beta}$ exist and are ergodic.
The problem is to show the completeness of classification.
See [Gorin–Kerov–Vershik '12].

Random triangular matrices over a finite field

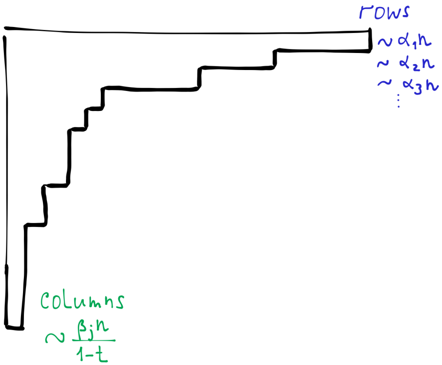
We construct a randomized RSK to sample these ergodic measures. Input of the RSK is a random Bernoulli word.

Using this RSK, we prove another conjecture of Vershik–Kerov — a law of large numbers for the measures $\mu_n^{\alpha;\beta}$ ($t = 0$ — infinite symmetric group)

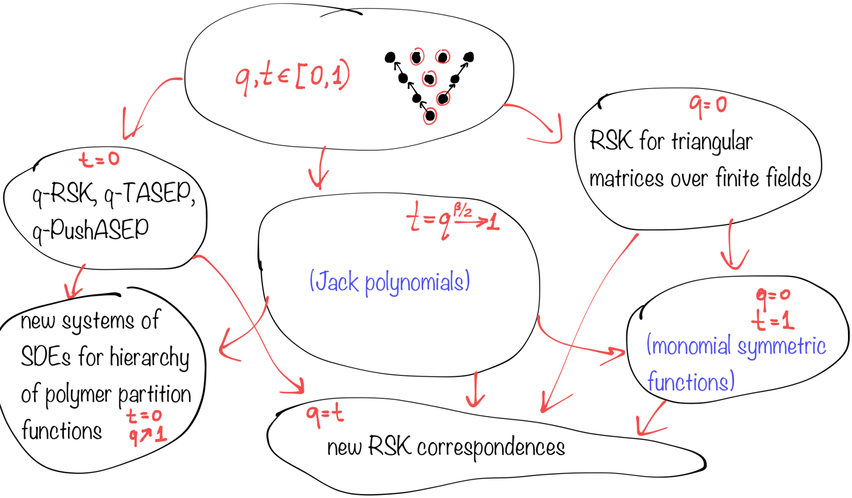
Theorem
[Bufetov–P., in progress].
For random Young diagrams distributed according to $\mu_n^{\alpha;\beta}$, as $n \to \infty$:

$$\frac{\mathrm{row}(i)}{n} \to \alpha_i$$

$$\frac{\mathrm{column}(j)}{n} \to \frac{\beta_j}{1-t}$$



Conclusion



$q,t \in [0,1)$
$q=0$
RSK for triangular matrices over finite fields
$t=0$
q-RSK, q-TASEP, q-PushASEP
$t=q^{\beta/2} \to 1$
(Jack polynomials)
$q=0$
$t=1$
(monomial symmetric functions)
new systems of SDEs for hierarchy of polymer partition functions
$t=0$
$q \nearrow 1$
$q=t$
new RSK correspondences