# Calculus of conformal fields on a compact Riemann surface

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Recent developments in Constructive Field Theory

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# Outline

- Implementation of CFT constructed from GFF on a compact Riemann surface.
- Fields = certain types of Fock space fields + tensor nature.
  Cf. Gaussian free field and conformal field theory, Astérisque, 353 (2013).
- We treat a stress tensor in terms of Lie derivatives.
- Ward's equation and its examples $(g=1)$:
  - Addition theorem of Weierstrass $\wp$-function;
  - $\wp\wp' = \dfrac{1}{12}\wp'''$.
- Eguchi-Ooguri equation $(g=1)$: for any tensor product $\mathcal{X}$ of fields in the OPE family $\mathcal{F}$, in the $\mathbb{T}_\Lambda$-uniformization

$$\frac{1}{2\pi i}\oint_{[0,1]} \mathbf{E}\, A(\xi)\mathcal{X}\, \mathrm{d}\xi = \frac{\partial}{\partial\tau}\mathbf{E}\,\mathcal{X}.$$

- Eguchi-Ooguri's version of Ward's equation

$$\mathbf{E}\, A(\xi)\mathcal{X} = \mathcal{L}^+_{\tilde{v}_\xi}\mathbf{E}\,\mathcal{X} + 2\pi i\,\frac{\partial}{\partial\tau}\mathbf{E}\,\mathcal{X}, \quad (\tilde{v}_\xi(z) = \zeta(\xi - z) + 2\eta_1 z).$$

  T. Eguchi and H. Ooguri. Conformal and current algebras on a general Riemann surface. *Nuclear Phys.* B, 282(2):308–328, 1987.

# Related Topics

- CFT with ($c \leq 1$) constructed from the central/background charge modifications of GFF.
  - The background charge of (simple) PPS form $\varphi$ is given by

$$\boldsymbol{\beta} = \frac{i}{\pi}\,\partial\bar{\partial}\varphi = \sum \beta_j \delta_{q_j}$$

  with the *neutrality condition*,

$$\int \boldsymbol{\beta} (= \sum \beta_k) = b\chi(M), \qquad c = 1 - 12b^2.$$

- (With Byun & Tak) Implementation of CFT in a doubly connected domain.
  - Dirichlet boundary condition and ER (Excursion Reflected) boundary condition.
  - The neutrality condition: total sum of background charges is zero.
  - A connection to annulus SLE theory.

# Gaussian free field

The Gaussian free field $\Phi$ on a compact Riemann surface $M$ is a Gaussian field indexed by the energy space $\mathcal{E}(M)$,

$$\Phi : \mathcal{E}(M) \to L^2(\Omega, \mathbf{P});$$

here $(\Omega, \mathbf{P})$ is some probability space. By definition, $\Phi$ is an isometry such that the image consists of centered Gaussian random variables.

The energy space $\mathcal{E} = \mathcal{E}(M)$ is the completion of test functions $f$ satisfying

$$\int f = 0$$

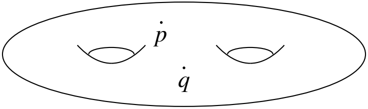
with respect to

$$\|f\|_{\mathcal{E}}^2 = \iint 2G_{\zeta,\eta}(z) f(z) \overline{f(\zeta)}$$

for all $\eta \in M$.

# Bipolar Green's function

Let $p, q$ be distinct marked points of the compact Riemann surface $M$.




By definition, bipolar Green's function $z \mapsto G_{p,q}(z)$ with singularities at $p$ and $q$ is harmonic on $M \setminus \{p, q\}$, and satisfies

$$G_{p,q}(z) = \log \frac{1}{|z-p|} + O(1) \qquad (z \to p),$$
$$G_{p,q}(z) = -\log \frac{1}{|z-q|} + O(1) \qquad (z \to q)$$

(in some/any chart).

Note that a bipolar Green's function is not uniquely determined. However, it is unique up to adding a constant.

# Gaussian free field

We introduce the Fock space functionals $\Phi(z, z_0)$ as "generalized" elements of Fock space

$$\Phi(z, z_0) = \Phi(\delta_z - \delta_{z_0}),$$

where $\delta_z - \delta_{z_0}$ is the "generalized" elements of $\mathcal{E}(M)$.

We now define the correlation function of Gaussian free field by

$$\mathbf{E}[\Phi(p,q)\Phi(\tilde{p},\tilde{q})] = 2(G_{p,q}(\tilde{p}) - G_{p,q}(\tilde{q})), \quad (\tilde{p},\tilde{q} \notin \{p,q\}).$$

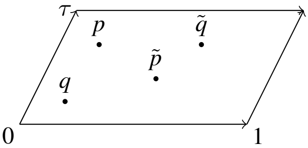
On the Riemann sphere,

$$\mathbf{E}\,\Phi(p,q)\Phi(\tilde{p},\tilde{q}) = \log|\lambda(p,q;\tilde{p},\tilde{q})|^2,$$

where

$$\lambda(p,q;\tilde{p},\tilde{q}) = \frac{(\tilde{p}-q)(\tilde{q}-p)}{(\tilde{p}-p)(\tilde{q}-q)}.$$

# Gaussian free field




On the Torus $\mathbb{T}_\Lambda = \mathbb{C}/\Lambda$, $(\Lambda = \mathbb{Z} + \tau\mathbb{Z}, \mathrm{Im}\ \tau > 0)$,

$$\mathbf{E}\,\Phi(p,q)\Phi(\tilde{p},\tilde{q}) = \log|\lambda(p,q;\tilde{p},\tilde{q})|^2 - 4\pi\,\frac{\mathrm{Im}(p-q)\ \mathrm{Im}\,(\tilde{p}-\tilde{q})}{\mathrm{Im}\ \tau},$$

where

$$\lambda(p,q;\tilde{p},\tilde{q}) = \frac{\theta(\tilde{p}-q)\theta(\tilde{q}-p)}{\theta(\tilde{p}-p)\theta(\tilde{q}-q)}$$

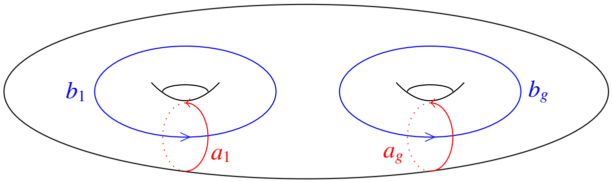
and

$$\theta(z) = \theta(z\,|\,\tau) = 2\sum_{n=1}^{\infty}(-1)^{n-1}e^{\pi i\tau(n-\frac{1}{2})^2}\sin(2n-1)\pi z.$$

# Canonical basis

Let $M$ be a compact Riemann surface of genus $g \geq 1$.

Fix a canonical basis $\{a_j, b_j\}$ for the homology $H_1 = H_1(M)$ with the following intersection properties: $a_j \cdot b_j = 1$ and all other intersection numbers are zero.



# Period matrix

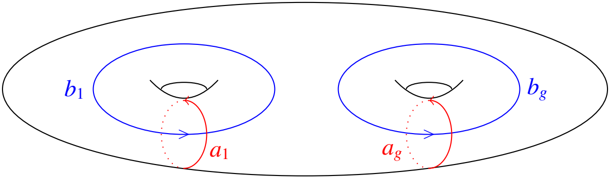
Let $\Omega(M)$ be the space of all holomorphic 1-differentials on $M$ and let $\{\omega_j\}$ be its basis uniquely determined by the equations

$$\oint_{a_k} \omega_j = \delta_{jk}.$$

The period matrix $\tau = \{\tau_{jk}\}$ is defined as

$$\tau_{jk} = \oint_{b_k} \omega_j.$$

The period matrix is symmetric and its imaginary part is positive definite, Im $\tau > 0$.



# Theta function

Let $\tau$ be a symmetric $g \times g$ matrix with Im $\tau > 0$ (e.g., the period matrix of a Riemann surface). The theta function $\Theta(\cdot \,|\, \tau)$ associated to $\tau$ is the following function of $g$ complex variables $Z = (z_1, \cdots, z_g)$

$$\Theta(Z \,|\, \tau) = \sum_{N \in \mathbb{Z}^g} e^{2\pi i (Z \cdot N + \frac{1}{2} \tau N \cdot N)} \qquad (Z \in \mathbb{C}^g).$$

The theta function is an *even* entire function on $\mathbb{C}^g$ (or a multivalued function on the Jacobi variety). It has the following periodicity properties: For $N \in \mathbb{Z}^g$, we have

$$\Theta(Z + N) = \Theta(Z), \qquad \Theta(Z + \tau N) = e^{-2\pi i (Z \cdot N + \frac{1}{2} \tau N \cdot N)} \Theta(Z).$$

## Gaussian free field

We consider the lattice $\Lambda = \mathbb{Z}^g + \tau\mathbb{Z}^g$ in $\mathbb{C}^g$ associated to the period matrix $\tau$ of $M$ and set

$$\mathbb{T}_\Lambda \equiv \mathbb{T}_\Lambda^g := \mathbb{C}^g/\Lambda.$$

Then

$$\mathbf{E}\,\Phi(p,q)\Phi(\tilde{p},\tilde{q}) = \log|\lambda(p,q;\tilde{p},\tilde{q})|^2 - 4\pi(\mathrm{Im}\ \tau)^{-1}\,\mathrm{Im}(P-Q)\cdot\mathrm{Im}\,(\tilde{P}-\tilde{Q}),$$

where

$$P - Q = \mathcal{A}(p) - \mathcal{A}(q) = \int_q^p \vec{\omega}, \quad \vec{\omega} = (\omega_1, \cdots, \omega_g)$$

and

$$\lambda(p,q;\tilde{p},\tilde{q}) = \frac{\theta(\tilde{p}-q)\theta(\tilde{q}-p)}{\theta(\tilde{p}-p)\theta(\tilde{q}-q)}, \quad \theta = \Theta\circ\mathcal{A}.$$

Cf. In the $g=1$ case,

$$\mathbf{E}\,\Phi(p,q)\Phi(\tilde{p},\tilde{q}) = \log|\lambda(p,q;\tilde{p},\tilde{q})|^2 - 4\pi\,\frac{\mathrm{Im}(p-q)\ \mathrm{Im}\,(\tilde{p}-\tilde{q})}{\mathrm{Im}\ \tau}.$$

# Fock space fields

*Fock space fields* are obtained from the Gaussian free field (GFF) $\Phi$ by applying the basic operations:

i. derivatives;
ii. Wick's products;
iii. multiplying by scalar functions and taking linear combinations.

**Examples**

$$J = \partial\Phi, \quad \Phi\odot\Phi(\equiv :\Phi\Phi:), \quad J\odot\Phi, \quad J\odot J, \quad e^{\odot\alpha\Phi} = \sum_{n=0}^{\infty}\frac{\alpha^n\Phi^{\odot n}}{n!}, \quad e^{\odot\beta J}.$$

**Examples**

- $\mathbf{E}[J(\zeta)J(z)] = \partial_\zeta\partial_z\mathbf{E}[\Phi(\zeta,\zeta_0)\Phi(z,z_0)]$.
- $J(\zeta)\odot J(z) = J(\zeta)J(z) - \mathbf{E}[J(\zeta)J(z)]$.

# OPE

We write the OPE of two (*holomorphic*) fields $X(\zeta)$ and $Y(z)$ as

$$X(\zeta)Y(z) = \sum C_j(z)(\zeta - z)^j \quad (\zeta \to z, \zeta \neq 0).$$

Write $X * Y$ for $C_0$.

**Example** ($g = 1$) We have

$$J(\zeta)J(z) = \mathbf{E}[J(\zeta)J(z)] + J(\zeta) \odot J(z).$$

In the identity chart of $\mathbb{T}_\Lambda$,

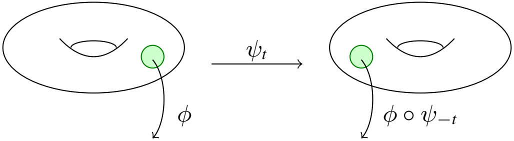
$$\mathbf{E}\, J(\zeta)J(z) = \partial_z\Big( - \frac{\theta'(\zeta - z)}{\theta(\zeta - z)} + \frac{\pi}{\text{Im}\ \tau} z\Big) = -\wp(\zeta - z) + \frac{1}{3}\frac{\theta'''(0)}{\theta'(0)} + \frac{\pi}{\text{Im}\ \tau},$$

where $\wp(z) := \dfrac{1}{z^2} + \sum_{m,n}{}'\big(\dfrac{1}{(z+m+n\tau)^2} - \dfrac{1}{(m+n\tau)^2}\big).$

As $\zeta \to z$, $\mathbf{E}\, J(\zeta)J(z) = -\dfrac{1}{(\zeta - z)^2} + \dfrac{1}{3}\dfrac{\theta'''(0)}{\theta'(0)} + \dfrac{\pi}{\text{Im}\ \tau} + o(1)$. In $\text{id}_{\mathbb{T}_\Lambda}$,

$$J * J = J \odot J + \frac{1}{3}\frac{\theta'''(0)}{\theta'(0)} + \frac{\pi}{\text{Im}\ \tau}.$$

# Lie derivative




Let

$$(X_t \parallel \phi)(z) = (X \parallel \phi \circ \psi_{-t})(z),$$

where $\psi_t$ is a local flow of $v$.

We define the Lie derivative (or fisherman's derivative) of $X$ by

$$(\mathcal{L}_v X \parallel \phi)(z) = \frac{d}{dt}\bigg|_{t=0} (X \parallel \phi \circ \psi_{-t})(z).$$

The flow carries all possible differential geometric objects past the fisherman, and the fisherman sits there and differentiates them.

Cf. V. I. Arnold, *Mathematical Methods of Classical Mechanics*.

# Lie derivative

If $X$ is a differential, then

$$X_t(z) = (X(\psi_t z) \parallel \psi_{-t}) = (\psi_t'(z))^{\lambda} \left(\overline{\psi_t'(z)}\right)^{\lambda_*} X(\psi_t z);$$

and

$$\mathcal{L}_v X = \left(v\partial + \lambda v' + \bar{v}\bar{\partial} + \lambda_* \overline{v'}\right) X.$$

The Lie derivative operator $v \mapsto \mathcal{L}_v$ depends $\mathbb{R}$-linearly on $v$. Denote

$$\mathcal{L}_v^+ = \frac{\mathcal{L}_v - i\mathcal{L}_{iv}}{2}, \qquad \mathcal{L}_v^- = \frac{\mathcal{L}_v + i\mathcal{L}_{iv}}{2},$$

so that

$$\mathcal{L}_v = \mathcal{L}_v^+ + \mathcal{L}_v^-.$$

If $X$ is a differential, then

$$\mathcal{L}_v^+ X = \left(v\partial + \lambda v'\right) X.$$

# Stress tensor

- A pair of quadratic differentials $W = (A_+, A_-)$ is called a stress tensor for $X$ if "residue form of Ward's identity" holds:

$$\mathcal{L}_v^+ X(z) = \frac{1}{2\pi i} \oint_{(z)} v A_+ X(z)$$
$$\mathcal{L}_v^- X(z) = -\frac{1}{2\pi i} \oint_{(z)} \bar{v} A_- X(z),$$

where $\mathcal{L}_v^{\pm} = \dfrac{\mathcal{L}_v \mp i\mathcal{L}_{iv}}{2}$.

Notation: $\mathcal{F}(W)$ is the family of fields with stress tensor $W = (A_+, A_-)$.

- If $X, Y \in \mathcal{F}(W)$, then $\partial X, X * Y \in \mathcal{F}(W)$.

# Stress tensor

- We have a stress tensor

$$W = (A, \bar{A}), \quad A = -\frac{1}{2} J \odot J$$

for $\Phi$ and its OPE family.

- **Example.**

$$A = -\frac{1}{2} J \odot J \notin \mathcal{F}(W), \text{ but } T = -\frac{1}{2} J * J = -\frac{1}{2} J \odot J + \frac{1}{12} S \in \mathcal{F}(W),$$

where

$$S(z) = 12\, \mathbf{E}\, T(z) = 6 \lim_{\zeta \to z} \Big( -2 \partial_\zeta \partial_z G_{z,z_0}(\zeta) - \frac{1}{(\zeta - z)^2} \Big)$$

is a Schwarzian form of order 1.

# Ward's equation

Given a meromorphic vector field $v$ with poles $\eta_1, \ldots, \eta_N$, we define the Ward functional $W^+$ by

$$W^+(v) = -\frac{1}{2\pi i} \sum \oint_{(\eta_k)} vA.$$

## Theorem (K.-Makarov)

*If $X_j$'s are in $\mathcal{F}(A, \bar{A})$, then*

$$\mathbf{E}\, W^+(v)\mathcal{X} = \mathbf{E}\, \mathcal{L}_v^+ \mathcal{X}.$$

## Corollary

*If $X_j$'s are primary fields with conformal dimensions $(\lambda_j, \lambda_{*j})$, then in the usual uniformization,*

$$-\sum \mathbf{E}\ \mathrm{Res}_{\eta_k}(vA)\mathcal{X} = \sum (v(z_j)\partial_j + \lambda_j v'(z_j))\mathbf{E}\, \mathcal{X},$$

*where* $\mathcal{X} = X_1(z_1) \cdots X_n(z_n)$.

# Ward's equation ($g = 1$)

On the torus $\mathbb{T}_\Lambda$, with the choice of $v_{\eta,\eta_0}(z) = \dfrac{\theta'}{\theta}(\eta - z) - \dfrac{\theta'}{\theta}(\eta_0 - z)$,

$$\mathbf{E}\,(A(\eta) - A(\eta_0))\mathcal{X} = \sum (v_{\eta,\eta_0}(z_j)\partial_j + \lambda_j v'_{\eta,\eta_0}(z_j))\mathbf{E}\,\mathcal{X}$$

for the tensor product $\mathcal{X}$ of primary fields with conformal dimensions $(\lambda_j, \lambda_{*j})$.

With the choice of $v(z) = v_\eta(z) = -\wp(\eta - z)$, we have

$$\mathbf{E}\,\partial A(\eta)\mathcal{X} = \sum (v_\eta(z_j)\partial_j + \lambda_j v'_\eta(z_j))\mathbf{E}\,\mathcal{X}.$$

Ward's equation with $\mathcal{X} = J(z)J(z_0)$ gives the addition theorem of Weierstrass $\wp$-function:

$$\begin{vmatrix} 1 & 1 & 1 \\ \wp(\eta - z) & \wp(z - z_0) & \wp(\eta - z_0) \\ \wp'(\eta - z) & \wp'(z - z_0) & -\wp'(\eta - z_0) \end{vmatrix} = 0.$$

Recall

$$\mathbf{E}\,J(\zeta)J(z) = -\wp(\zeta - z) + \frac{1}{3}\frac{\theta'''(0)}{\theta'(0)} + \frac{\pi}{\operatorname{Im}\,\tau},$$

# An example for Ward's equation

With the choice of $v(z) = v_\eta(z) = -\wp(\eta - z)$, we have

$$\mathbf{E}\,\partial A(\eta)\mathcal{X} = \mathcal{L}_{v_\eta}\mathbf{E}\,\mathcal{X},$$

for the tensor product of fields in the OPE family of $\Phi$.

Ward's equation with $\mathcal{X} = T(z) := -\frac{1}{2}J * J(z) = -\frac{1}{2}J \odot J(z) + \frac{1}{12}S$ gives

$$\begin{aligned}\partial_\eta \mathbf{E}\,A(\eta)T(z) &= \mathcal{L}_{v_\eta}\mathbf{E}\,\mathcal{X} = (v_\eta\partial + 2v_\eta')\mathbf{E}\,\mathcal{X} + \frac{1}{12}v_\eta''' \\ &= 2\,\mathbf{E}\,T\,\wp'(\eta - z) + \frac{1}{12}\wp'''(\eta - z).\end{aligned}$$

Applying Wick's calculus to the left-hand side,

$$\partial_\eta \mathbf{E}\,A(\eta)T(z) = \frac{1}{2}\partial_\eta(\mathbf{E}\,J(\eta)J(z))^2 = (\wp(\eta - z) + 2\,\mathbf{E}\,T)\wp'(\eta - z).$$

Thus we have

$$\wp\wp' = \frac{1}{12}\wp'''.$$

# Eguchi-Ooguri equation on a torus

## Theorem

*For any tensor product $\mathcal{X}$ of fields in the OPE family $\mathcal{F}$,*

$$\frac{1}{2\pi i}\oint_{[0,1]} \mathbf{E}\, A(\xi)\mathcal{X}\,\mathrm{d}\xi = \frac{\partial}{\partial\tau}\,\mathbf{E}\,\mathcal{X} \tag{1}$$

*in the $\mathbb{T}_\Lambda$-uniformization.*

## Theorem

*For any tensor product $\mathcal{X}$ of fields in the OPE family $\mathcal{F}$, we have*

$$\mathbf{E}\, A(\xi)\mathcal{X} = \mathcal{L}^{+}_{\tilde{v}_\xi}\mathbf{E}\,\mathcal{X} + 2\pi i\,\frac{\partial}{\partial\tau}\,\mathbf{E}\,\mathcal{X}, \quad (\tilde{v}_\xi(z) = \zeta(\xi - z) + 2\eta_1 z)$$

*in the $\mathbb{T}_\Lambda$-uniformization.*

# Weierstrass $\zeta$-function

Weierstrass $\zeta$-function

$$\zeta(z) := \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right)$$

is a meromorphic odd function which has simple poles at $\lambda \in \Lambda$ with residue 1.
Weierstrass $\zeta$-function has the following quasi-periodicities

$$\zeta(z+m+n\tau) = \zeta(z) + 2m\eta_1 + 2n\eta_2, \qquad \big(\eta_1 = \zeta(1/2), \quad \eta_2 = \zeta(\tau/2)\big).$$

Here are two main ingredients for the proof of Eguchi-Ooguri equation.

- Frobenius-Stickelberger's pseudo-addition theorem for Weierstrass $\zeta$-function:

  $$\big(\zeta(z_1)+\zeta(z_2)+\zeta(z_3)\big)^2 + \zeta'(z_1)+\zeta'(z_2)+\zeta'(z_3) = 0, \qquad (z_1+z_2+z_3=0),$$

  equivalently, by $\theta'(z)/\theta(z) = \zeta(z) - 2\eta_1 z$

  $$2\sum_{j<k} \frac{\theta'(z_j)}{\theta(z_j)} \frac{\theta'(z_k)}{\theta(z_k)} + \sum_j \frac{\theta''(z_j)}{\theta(z_j)} + 6\eta_1 = 0, \qquad (z_1+z_2+z_3=0).$$

- Jacobi-theta function satisfies the heat equation,

  $$2\pi i \frac{\partial}{\partial \tau} \theta = \frac{1}{2} \theta''.$$

# Eguchi-Ooguri equation: a sketch of proof

For $\mathcal{X} = \Phi(z, z_0)\Phi(z', z_0')$,

$$\mathbf{E}\,\mathcal{X} = -4\pi \frac{\mathrm{Im}(z - z_0)\ \mathrm{Im}\,(z' - z_0')}{\mathrm{Im}\ \tau} + \log\left|\frac{\theta(z' - z_0)\theta(z_0' - z)}{\theta(z' - z)\theta(z_0' - z_0)}\right|^2$$

By the heat equation $2\pi i \frac{\partial}{\partial\tau}\theta = \frac{1}{2}\theta''$,

$$2\pi i \frac{\partial}{\partial\tau}\mathbf{E}\mathcal{X} - 4\pi^2 \frac{\mathrm{Im}\,(z - z_0)\ \mathrm{Im}\,(z' - z_0')}{(\mathrm{Im}\ \tau)^2} = -\frac{1}{2}\left(\frac{\theta''(z - z')}{\theta(z - z')} + \cdots\right).$$

By the pseudo-addition theorem for $\zeta$,

$$\oint_{[0,1]} \mathbf{E}\,A(\xi)\mathcal{X}\,\mathrm{d}\xi - 4\pi^2 \frac{\mathrm{Im}\,(z - z_0)\ \mathrm{Im}\,(z' - z_0')}{(\mathrm{Im}\ \tau)^2}$$

$$= -\int_0^1 \frac{\theta'(\xi - z)}{\theta(\xi - z)}\frac{\theta'(\xi - z')}{\theta(\xi - z')}\,\mathrm{d}\xi + \frac{2\pi i}{\mathrm{Im}\ \tau}\left(\mathrm{Im}\ z \int_0^1 \frac{\theta'(\xi - z')}{\theta(\xi - z')}\,\mathrm{d}\xi + \cdots\right) + \cdots$$

$$= \frac{\theta'(z - z')}{\theta(z - z')}\int_0^1 \left(-\frac{\theta'(\xi - z)}{\theta(\xi - z)} + \frac{\theta'(\xi - z')}{\theta(\xi - z')}\right)\mathrm{d}\xi$$

$$-\frac{1}{2}\frac{\theta''(z - z')}{\theta(z - z')} - \frac{1}{2}\int_0^1 \frac{\theta''(\xi - z)}{\theta(\xi - z)}\,\mathrm{d}\xi - \frac{1}{2}\int_0^1 \frac{\theta''(\xi - z')}{\theta(\xi - z')}\,\mathrm{d}\xi - 6\eta_1 + \cdots .$$

# Eguchi-Ooguri equation: an example

We present a conformal field theoretic proof for

$$\eta_1 = -\frac{1}{6}\frac{\theta'''(0)}{\theta'(0)},$$

where $\eta_1 = \zeta(1/2)$.

For $\mathcal{X} = J(z)\overline{J(z)}$, it follows from Wick's formula that

$$\mathbf{E}\,A(\xi)\mathcal{X} = -\mathbf{E}\,J(\xi)J(z)\mathbf{E}\,J(\xi)\overline{J(z)} = -\Big(\wp(\xi - z) - \frac{1}{3}\frac{\theta'''(0)}{\theta'(0)} - \frac{\pi}{\mathrm{Im}\ \tau}\Big)\frac{\pi}{\mathrm{Im}\ \tau}.$$

Since $\wp = -\zeta'$ and $\zeta(z+1) = \zeta(z) + 2\eta_1$, we have

$$\oint_{[0,1]} \mathbf{E}\,A(\xi)\mathcal{X}\,\mathrm{d}\xi = \Big(2\eta_1 + \frac{1}{3}\frac{\theta'''(0)}{\theta'(0)}\Big)\frac{\pi}{\mathrm{Im}\ \tau} + \Big(\frac{\pi}{\mathrm{Im}\ \tau}\Big)^2.$$

On the other hand, we have

$$2\pi i\,\partial_\tau\,\mathbf{E}\,\mathcal{X} = -2\pi i\frac{\partial}{\partial\tau}\frac{\pi}{\mathrm{Im}\ \tau} = \Big(\frac{\pi}{\mathrm{Im}\ \tau}\Big)^2.$$

# Eguchi-Ooguri's version of Ward's equation on a torus: an example

## Theorem

*For any tensor product $\mathcal{X}$ of fields in the OPE family $\mathcal{F}$, we have*

$$\mathbf{E}\,A(\xi)\mathcal{X} = \mathcal{L}^{+}_{\tilde{v}_\xi}\mathbf{E}\,\mathcal{X} + 2\pi i\,\frac{\partial}{\partial\tau}\,\mathbf{E}\,\mathcal{X}, \quad (\tilde{v}_\xi(z) = \zeta(\xi - z) + 2\eta_1 z)$$

*in the $\mathbb{T}_\Lambda$-uniformization.*

**Example.** Applying $\mathcal{X} = J(z)J(z_0)$ $(z \neq z_0)$ to the above, we obtain

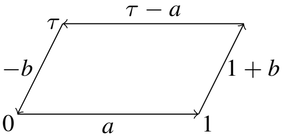
$$\frac{1}{2}\wp''(z_3) = \big(\wp(z_1) - \wp(z_3)\big)\big(\wp(z_2) - \wp(z_3)\big) + \frac{1}{2}\,\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\,\wp'(z_3)$$

if $z_1 + z_2 + z_3 = 0$.

# Eguchi-Ooguri's version of Ward's equation on $\mathbb{T}_\Lambda$: sketch of proof

Let us consider two vector fields $\tilde{v}^1(z) = z, \tilde{v}^2_\xi(z) = \zeta(\xi - z)$. We remark that these two vector fields have jump discontinuities across the cycles $a = [0, 1], b = [0, \tau]$. Consider a parallelogram $D := \{z \in \mathbb{C} \,|\, z = x + y\tau, x, y \in [0, 1]\}$ and use Green's formula

$$
\begin{aligned}
&-\frac{1}{\pi}\int_D \bar{\partial}\tilde{v}_\xi(z)\mathbf{E}\,A(z)\mathcal{X} \\
&\quad = \sum \frac{1}{2\pi i}\oint_{(z_j)} \tilde{v}_\xi(z)\mathbf{E}\,A(z)\mathcal{X}\,\mathrm{d}z \\
&\quad + \frac{\eta_1\tau - \eta_2}{\pi i}\oint_a \mathbf{E}\,A(z)\mathcal{X}\,\mathrm{d}z,
\end{aligned}
$$




where $\tilde{v}_\xi = \tilde{v}^2_\xi + 2\eta_1\tilde{v}^1$, so $\tilde{v}_\xi(z+1) = \tilde{v}_\xi(z)$.

- Since $\bar{\partial}\tilde{v}_\xi = -\pi\delta_\xi$, LHS $= \mathbf{E}\,A(\xi)\mathcal{X}$.
- By Ward's equation, RHS$_1$ $= \mathcal{L}^+_{\tilde{v}_\xi}\mathbf{E}\,\mathcal{X}$.
- Due to Legendre, $\eta_1\tau - \eta_2 = \pi i$.
- By Eguchi-Ooguri equation, RHS$_2$ $= 2\pi i\partial_\tau\,\mathbf{E}\,\mathcal{X}$.

Thank you very much.