Gauge theory and the three barriers

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PLAN

A Review Yang Mills and variants

B Review stories about matrix models and planar maps

C Discuss stories about embedded surfaces, loops and growth

D What are the "barriers" to a continuum theory?

Yang-Mills

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- $N = \infty$, $d = 0$ corresponds to pure LQG (Brownian map, etc.) in some sense.

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- Related stories: σ_N is GUE or GOE or Ginibre ensemble...

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- So is a Yang-Mills plaquette trace a degree four or a degree eight polynomial in the matrix entries?... Both it seems.

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How far is unitary matrix from identity?

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- In continuum theory, imagine you have (for each coordinate dimension) continuous function from space to Lie algebra. Curvature components have two parts: one involving derivatives, one involving a Lie bracket. Norm squared has degree four terms (making theory non-Gaussian).

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- In two dimensions, gauge fixing simplifies problem tremendously. Two dimensions can be place to test theories believed to hold in general dimension.

Early string theory work motivated by gauge theory

There are methods and formulae in science, which serve as **master-keys** to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling **sums over random surfaces**. These sums replace the old-fashioned (and extremely useful) **sums over random paths**. The replacement is necessary, because today **gauge invariance** plays the central role in physics. Elementary excitations in gauge theories are formed by the **flux lines** (closed in the absence of charges) and the time development of these lines forms the **world surfaces**. All transition amplitude[s] are given by the sums over all possible surfaces with fixed boundary. (A.M. Polyakov, Moscow, 1981.) [Pol81a]

Classical matrix-map stories

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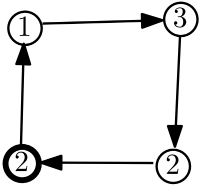

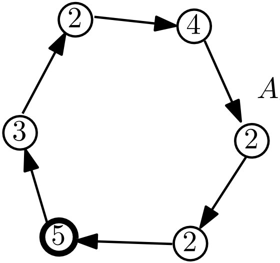

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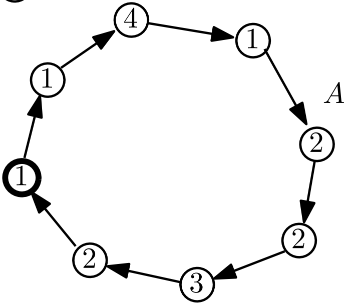

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- What if you have more than one matrix?

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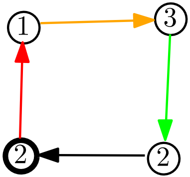

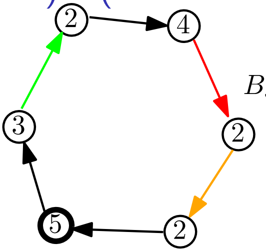

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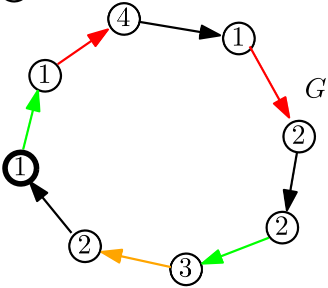

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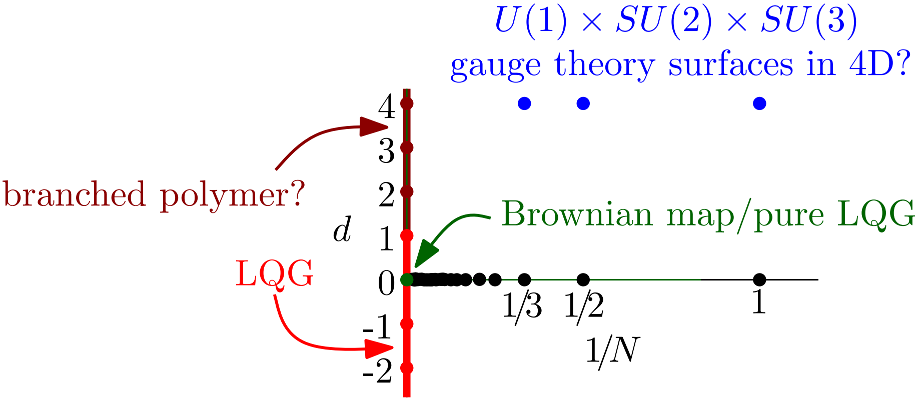
- There are many variants of this construction, which relate some kind of gauge theory to some kind of random surface model. A common theme is that the surfaces are embedded in the lattice, and that there is some weighting according to genus, depending on N.

Continuum scaling limits of random surfaces?



Can interpret d as a lattice dimension **or** (as we will later see) weight factor for planar maps (based on determinant of Laplacian). Can interpret N as a matrix dimension **or** as a weight factor (based on surface genus). Non-integer values of d and N make sense. But do we need a third dimension to deal with oscillatory weighting (where weight assigned to surface is e^{iK} where K is surface size, say)?

Background: determinant of discrete Laplacian

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- The DGFF partition function can be be written $\int_R (2\pi)^{-|V-1|/2} e^{-(f,-\Delta f)/2} df = \alpha^{-1/2}$.

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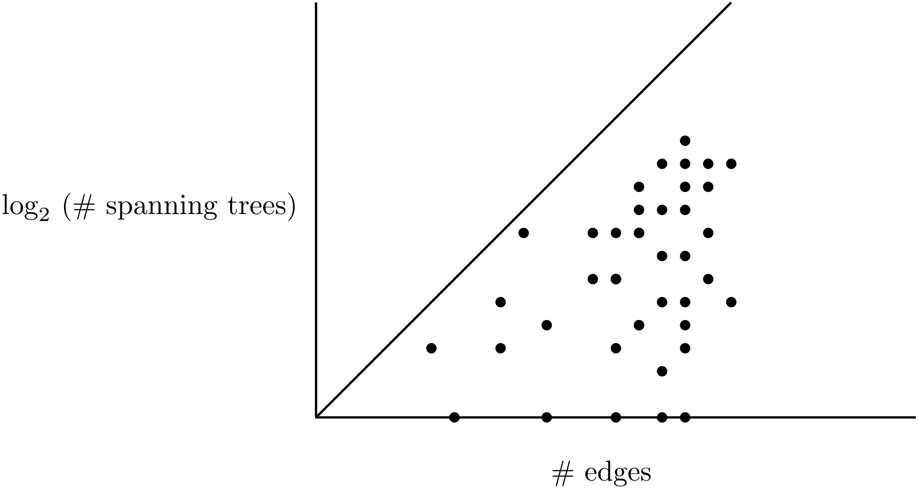
- A Poisson point process from measure with total mass $-\log \alpha$ can be said to have partition function α^{-1}. Multiplying intensity by constant changes power. Loop soups (of different intensities) have partition functions that are powers of $\det \Delta$.

Background: two measures of (sphere-embedded) planar map "size"



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- Fractional powers of determinant Laplacian describe loop soup decorations.
- In addition to weighting by determinant Laplacian powers, another way to interpolate involves the Tutte polynomial: namely, the FK cluster model partition function. Universality believed.

Fundamental question

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- Do properties of F imply existence of a random a.s. continuous function assigning trace values to loops?
- Is there some lovely continuum way to write $F(s)$ as a weighted sum over of loops spanning s?

How do we get Wilson expectations from random surfaces?

- As above, assign i.i.d. matrix to each edge of lattice. Compute $\mathbb{E}[e^{\beta \sum \mathbf{Tr} A}]$ with sum over A's obtained by multiplying around plaquettes. (Summing both clockwise and counterclockwise gives real part of trace.)

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- But let's think about what we can do by playing just with Wick's theorem.

More fun with Wick's theorem

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- What are the "surface soup" analogs of loop soups?
- What if I start with Ginibre-ensemble for each edge and weight edge by of $e^{-(\mathbf{Tr}(I-(AA^t)^m)^k}$ (or maybe sum over m values?) to make AA^t close to identity with high probability, so A is roughly unitary?

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- Why is this not a straightforward exercise?

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- What is special about the exponential function? How does it correspond to (formally) counting surfaces without labels?
- Can we compute $\langle W_{\ell_1} W_{\ell_2} \cdots W_{\ell_n} \rangle$ using just connected surfaces spanning $\ell_1, \ldots, \ell_n$?

Simple questions

- If h is discrete GFF, compute $\left\langle \prod_v (-\Delta h)^k \right\rangle$ in combinatorial terms. What if you had matrix valued discrete GFF?

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- How about $\left\langle [(AA^tAA^t)^n \right\rangle$? Consider (using Wishard distribution) what happens when weighting by AA^t and $-AA^tAA^t$. What can be said about limits?

Overview

Basic universal 2D random objects

1. **Universal random trees:** Brownian motion, continuum random tree
2. **Universal random surfaces:** quantum gravity, planar maps, string theory, CFT
3. **Universal random paths:** walks, interfaces, Schramm-Loewner evolution, CFT
4. **Universal random growth:** Eden model, DLA, DBM

Basic relationships

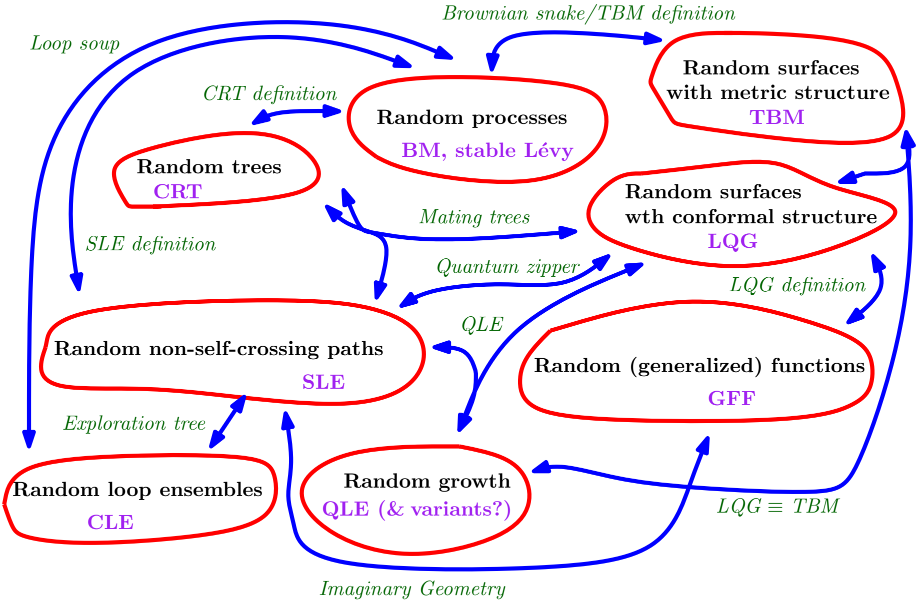
1. **Mating random trees:** tree plus tree (conformally mated) equals surface plus path
2. **Random growth on random surfaces:** dendrites, dragons, surprising tractability
3. **Mating random trees produced by a snake:** metric spaces and the Brownian map
4. **Two "universal random surfaces" are the same:** Brownian map equals Liouville quantum gravity with parameter $\gamma = \sqrt{8/3}$ (a.k.a. "pure quantum gravity").

Surfaces, strings, and matrix integrals

1. **Simple discrete story:** Spanning tree weighting versus GFF weighting
2. **Simple continuum story:** Dirichlet energy versus log Laplacian determinant
3. **Simple matrix story:** simplest GUE setting and variants
4. **Loop equations** What kinds of continuum process do we want?

Some random surface and SLE references

1. *Exploration trees and conformal loop ensembles* (S. 2006)
2. *Contour lines of the two-dimensional discrete GFF* (Schramm, S. 2006)
3. *Liouville quantum gravity and KPZ* (Duplantier, S. 2008)
4. *A contour line of the continuum Gaussian free field* (Schramm, S. 2008)
5. *Conformal Loop Ensembles: The Markovian characterization and the loop-soup construction* (S., Werner, 2010)
6. *Conformal weldings of random surfaces: SLE and the quantum gravity zipper* (S., 2010)
7. *Quantum gravity and inventory accumulation* (S., 2011)
8. *Imaginary Geometry I-IV* (Miller, S., 2012-2013)
9. *Quantum Loewner Evolution* (Miller, S. 2013)
10. *Liouville quantum gravity as a mating of trees* (Duplantier, Miller, S. 2014)
11. *Liouville quantum gravity spheres as matings of finite trees* (Miller, S 2015)
12. *An axiomatic characterization of the Brownian map* (Miller, S. 2015)
13. *Liouville quantum gravity and the Brownian map I-III* (Miller, S. 2015-2016)



Continuum objects and relationships (all have discrete analogs)

SOME UNIVERSAL FRIENDS

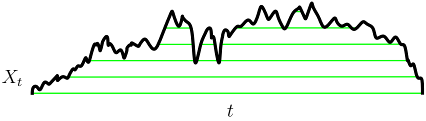
A Trees

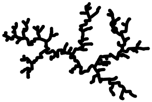
B Simple curves, non-simple curves, space-filling curves

C Surfaces

D Growth

RANDOM TREES





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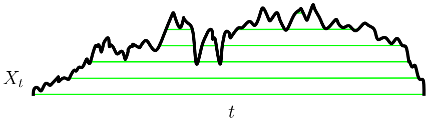

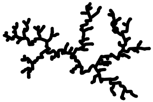

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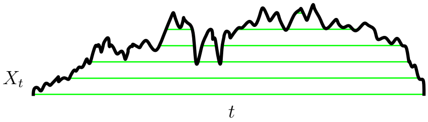

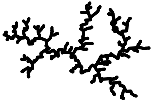

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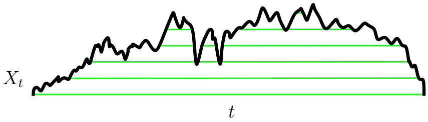

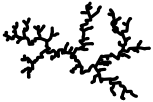

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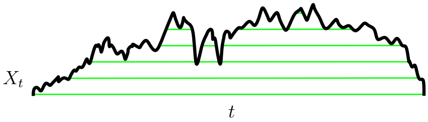

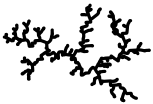

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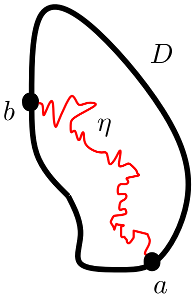
- Simple bijection between rooted planar trees and walks of this type.
- CRT is in some sense the "uniformly random planar tree" of a given size.

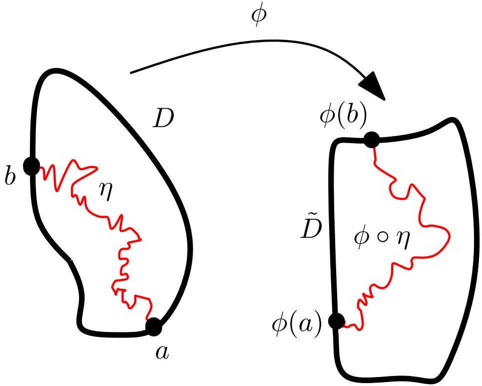
RANDOM PATHS

Given a simply connected planar domain D with boundary points a and b and a parameter $\kappa \in [0, \infty)$, the **Schramm-Loewner evolution** SLE_κ is a random non-self-crossing path in $\overline{D}$ from a to b.



The parameter κ roughly indicates how "windy" the path is. Would like to argue that SLE is in some sense the "canonical" random non-self-crossing path. What symmetries characterize SLE?

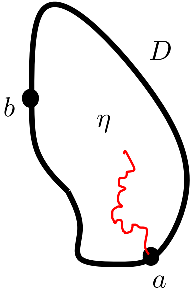
Conformal Markov property of SLE



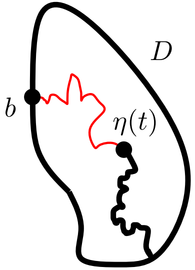
If ϕ conformally maps D to $\tilde{D}$ and η is an SLE$_\kappa$ from a to b in D, then $\phi \circ \eta$ is an SLE$_\kappa$ from $\phi(a)$ to $\phi(b)$ in $\tilde{D}$.

Markov Property

Given η up to a stopping time t...



law of remainder is SLE in $D \setminus \eta[0,t]$ from $\eta(t)$ to b.



Chordal Schramm-Loewner evolution (SLE)

- **THEOREM [Oded Schramm]:** Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$.

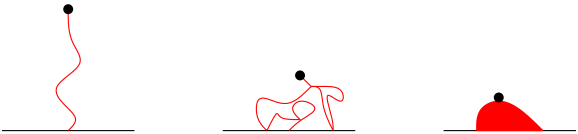
Chordal Schramm-Loewner evolution (SLE)

- **THEOREM [Oded Schramm]:** Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$.
- **Explicit construction:** An SLE path γ from 0 to ∞ in the complex upper half plane $\mathbf{H}$ can be defined in an interesting way: given path γ one can construct conformal maps $g_t : \mathbf{H} \setminus \gamma([0,t]) \to \mathbf{H}$ (normalized to look like identity near infinity, i.e., $\lim_{z\to\infty} g_t(z) - z = 0$). In SLE_κ, one defines g_t via an ODE (which makes sense for each fixed z):

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

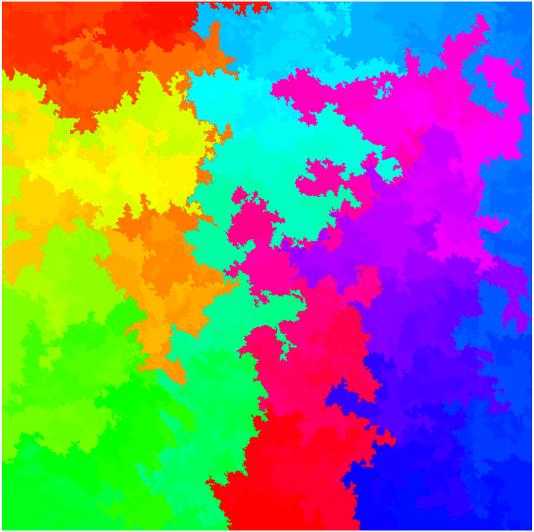
where $W_t = \sqrt{\kappa} B_t =_{LAW} B_{\kappa t}$ and B_t is ordinary Brownian motion.

SLE phases [Rohde, Schramm]

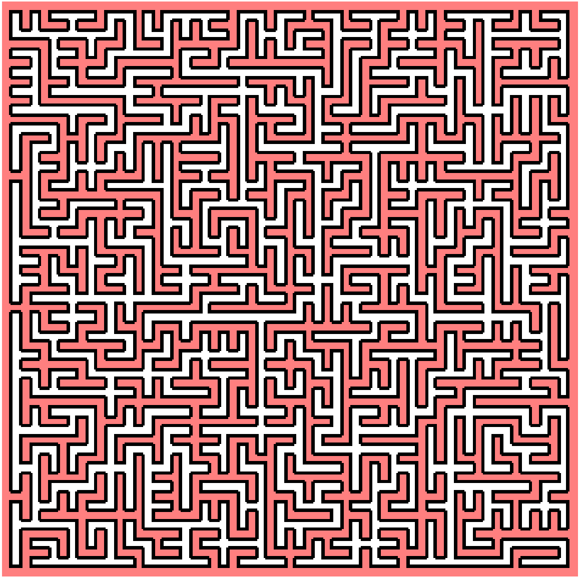


$\kappa \leq 4$ $\kappa \in (4,8)$ $\kappa \geq 8$

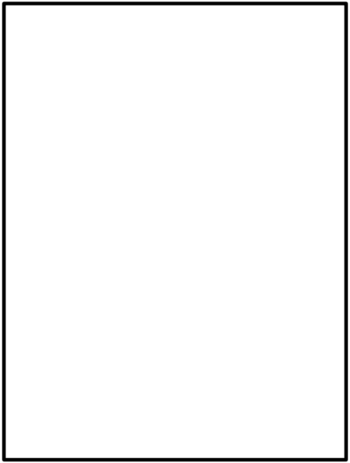
Continuum space-filling path



Uniform spanning tree

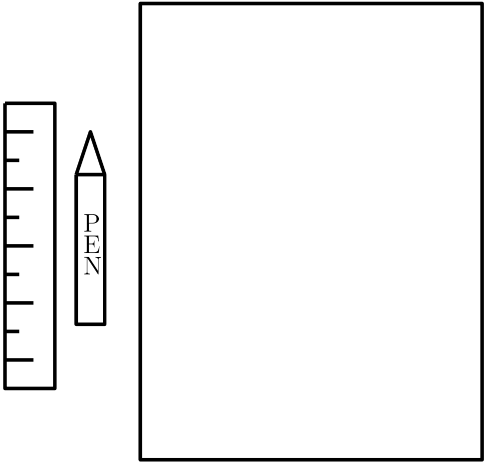


RANDOM SURFACES



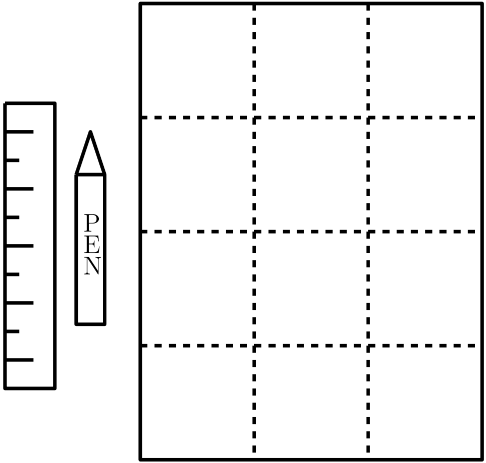
Start out with a sheet of paper

RANDOM SURFACES



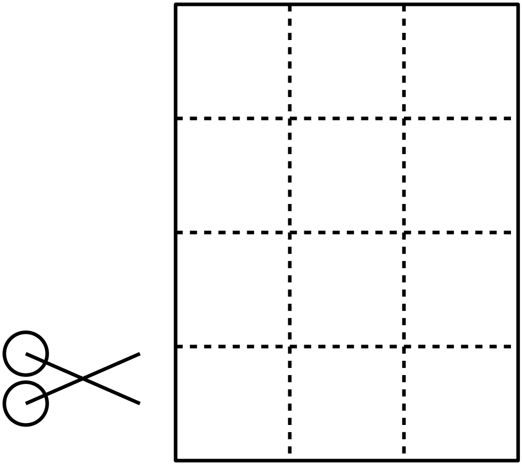
Get out pen and ruler

RANDOM SURFACES



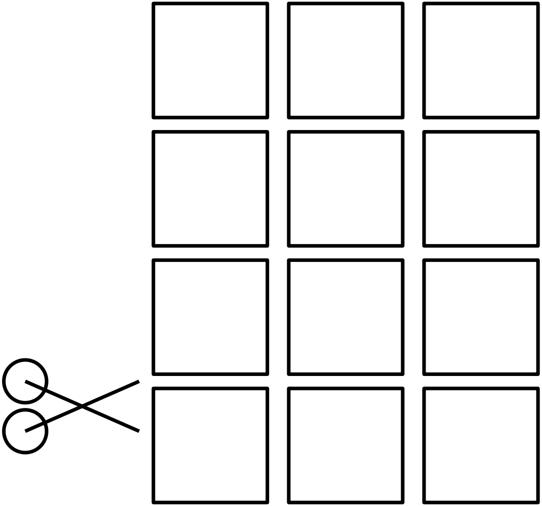
Measure and mark squares squares of equal size

RANDOM SURFACES



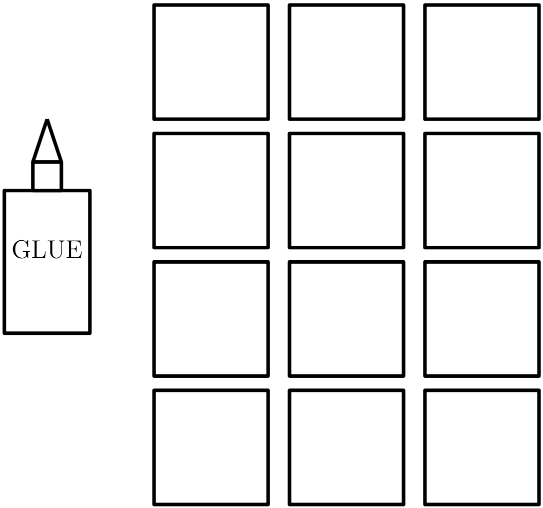
Get out scissors

RANDOM SURFACES



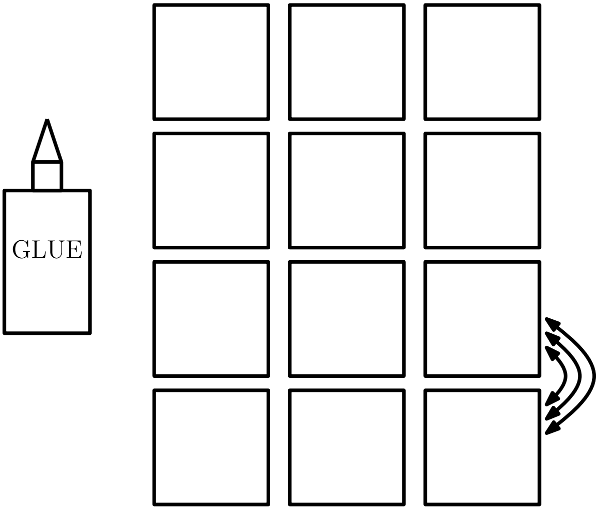
Cut into squares

RANDOM SURFACES

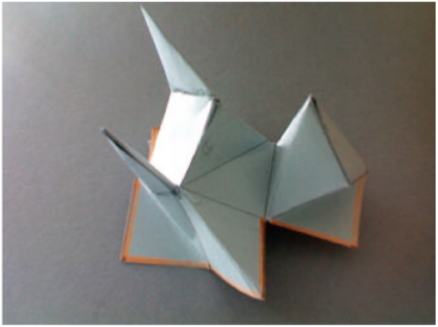


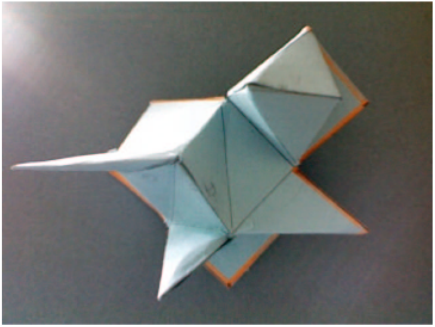
Get out bottle of glue

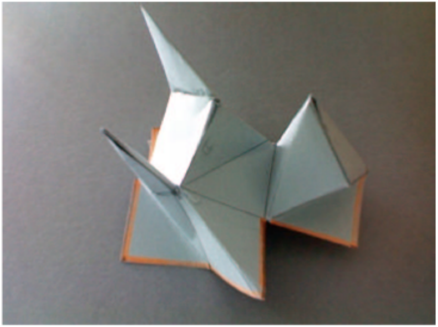
RANDOM SURFACES

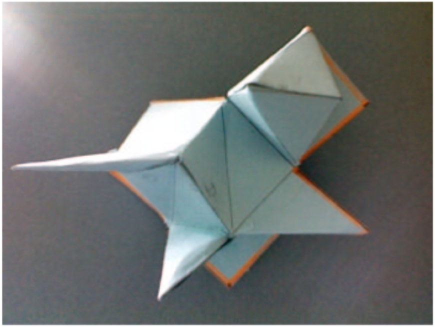


Attach squares along boundaries with glue to form a surface “without holes.”



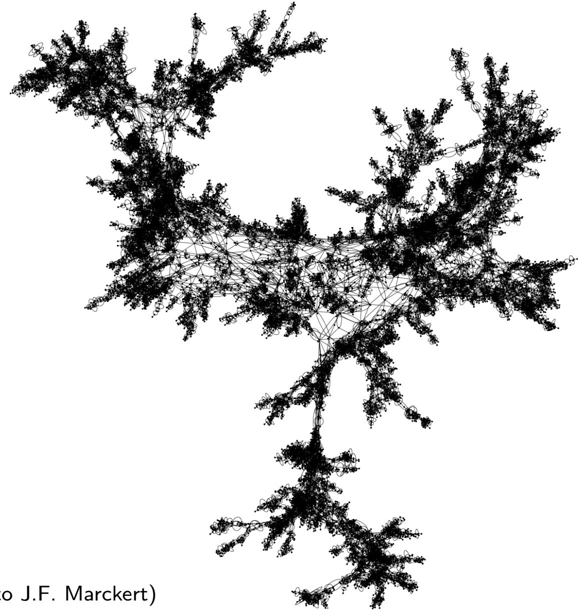






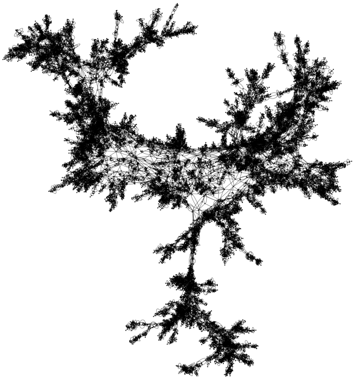
What is the structure of a typical quadrangulation when the number of faces is large?

Random quadrangulation with 25,000 faces



(Simulation due to J.F. Marckert)

Background



(Simulation due to J.F. Marckert)

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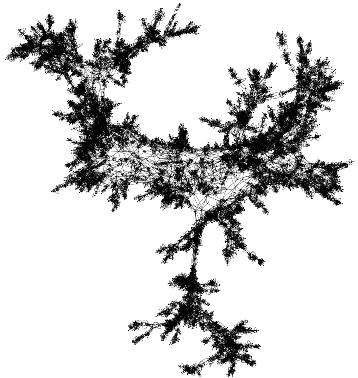

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2. Many variants (triangulations, quadrangulations, etc.) Some come equipped with extra statistical physics structure (a distinguished spanning tree, a general distinguished edge subset, a “spin” function on vertices, etc.)

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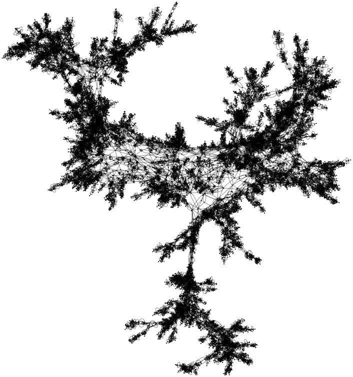

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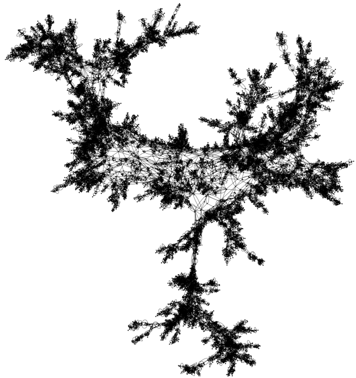

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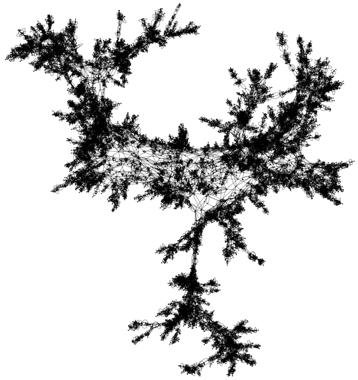

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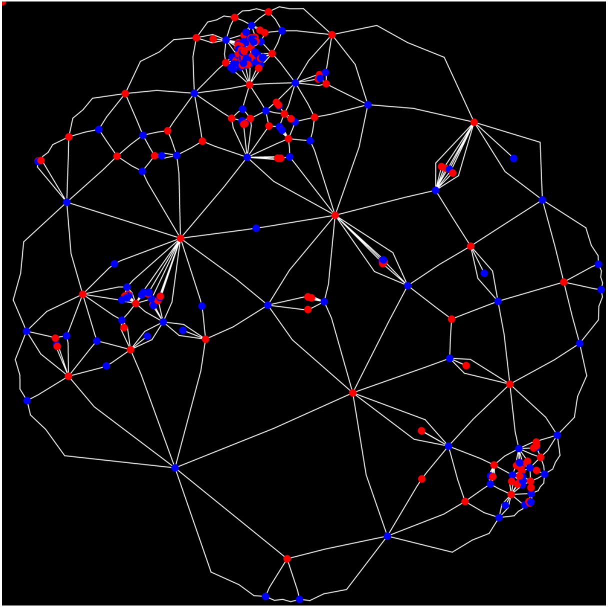
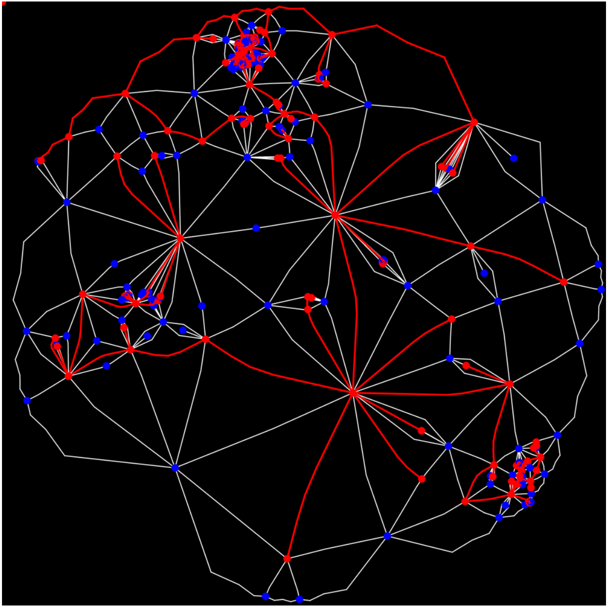
5. Important tool: Bijections encoding surface via pair of trees.

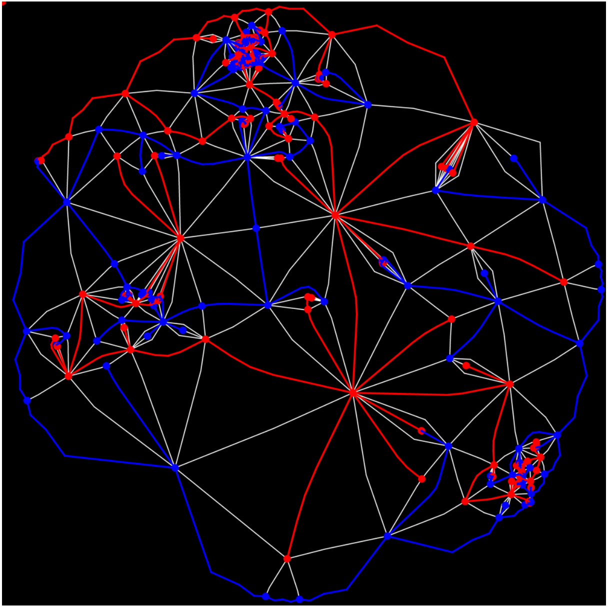
Random quadrangulation



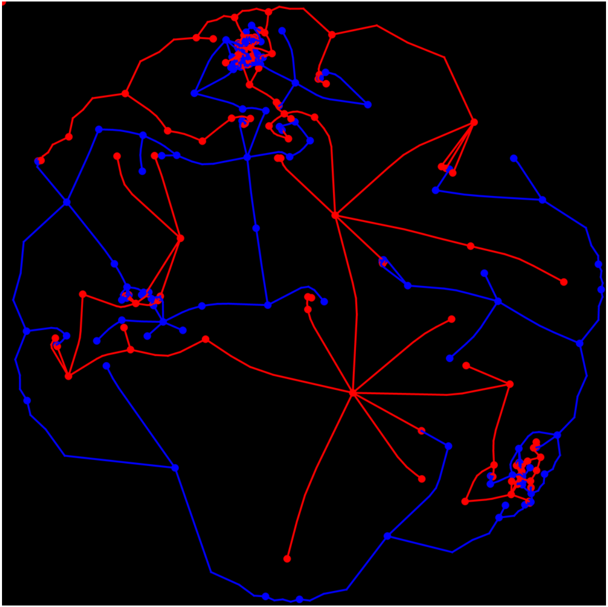
Red tree



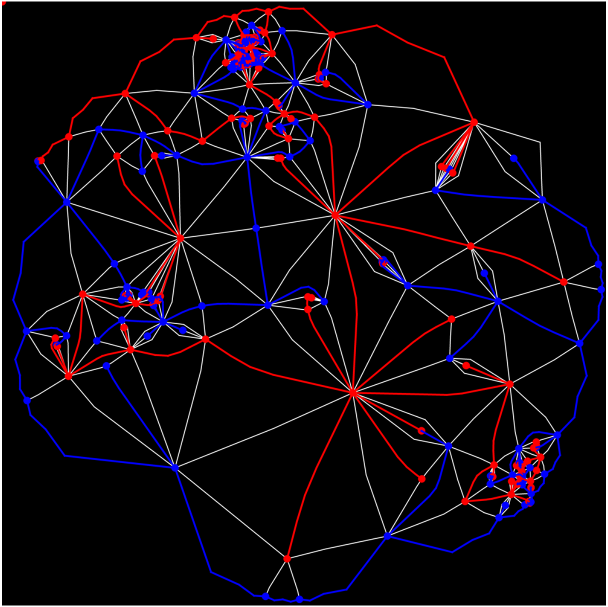
Red and blue trees



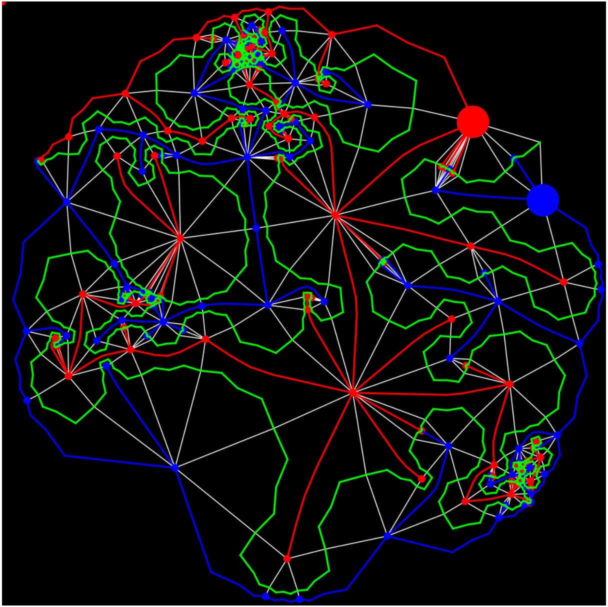
Red and blue trees alone do not determine the map structure



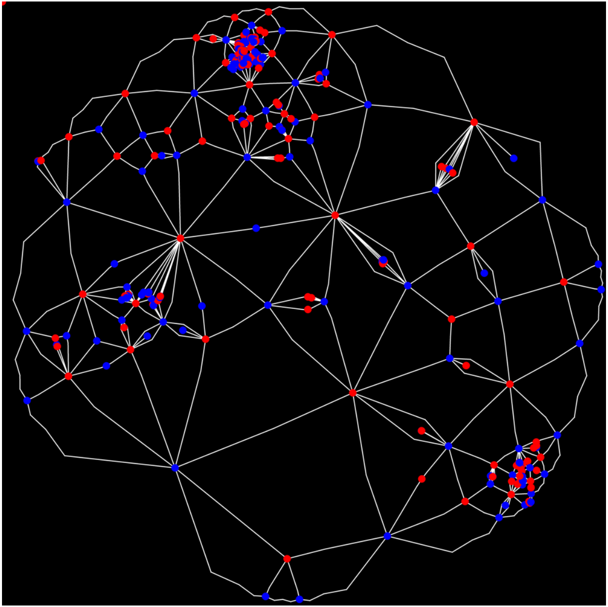
Random quadrangulation with red and blue trees



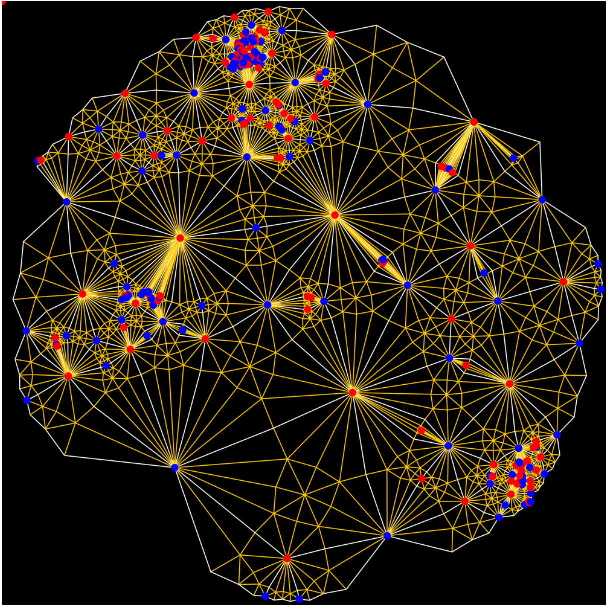
Path snaking between the trees. Encodes the trees and how they are glued together.



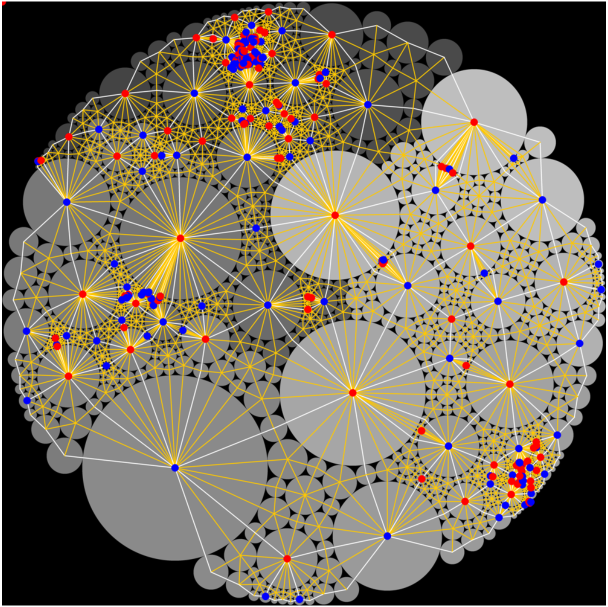
How was the graph embedded into $\mathbf{R}^2$?



Can subivide each quadrilateral to obtain a triangulation without multiple edges.

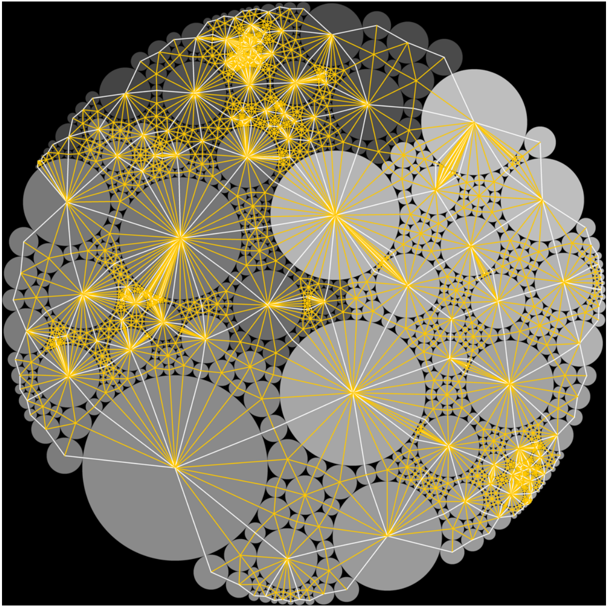


Circle pack the resulting triangulation.



Packed with Stephenson's CirclePack.

Circle pack the resulting triangulation.



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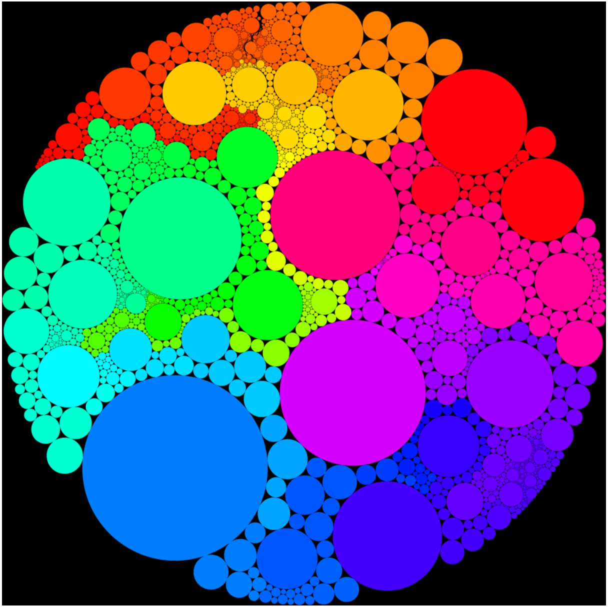
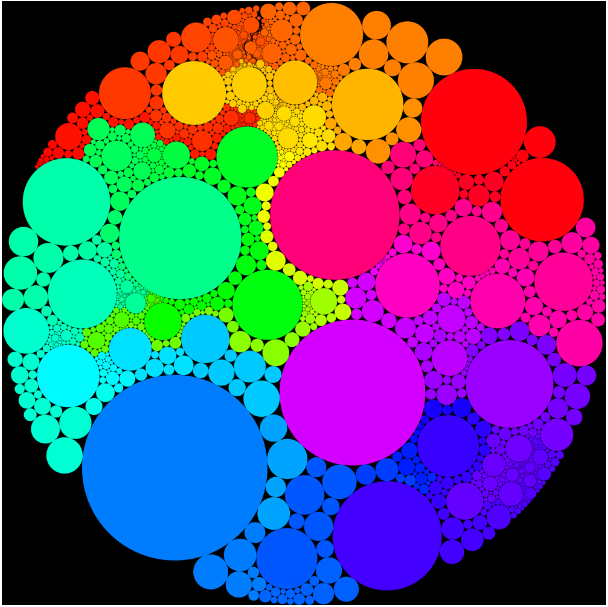


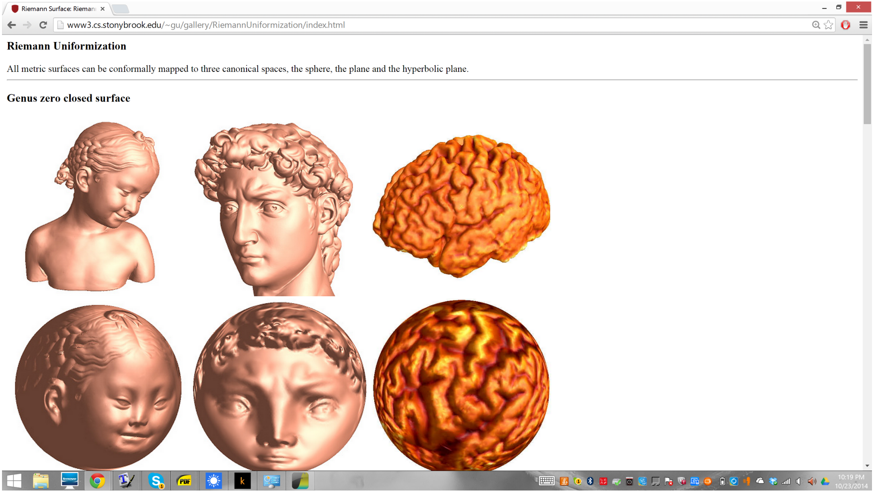
Packed with Stephenson's CirclePack.

What is the “limit” of this embedding? Circle packings are related to conformal maps.



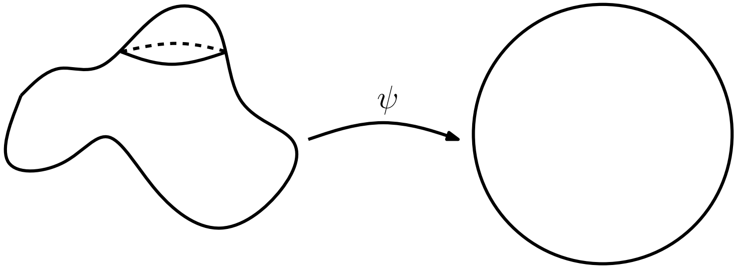
Packed with Stephenson’s CirclePack.

Conformal maps (from David Gu's web gallery)



Picking a surface at random in the continuum

Uniformization theorem: every simply connected Riemannian surface can be conformally mapped to either the unit disk, the plane, or the sphere $\mathbf{S}^2$ in $\mathbf{R}^3$



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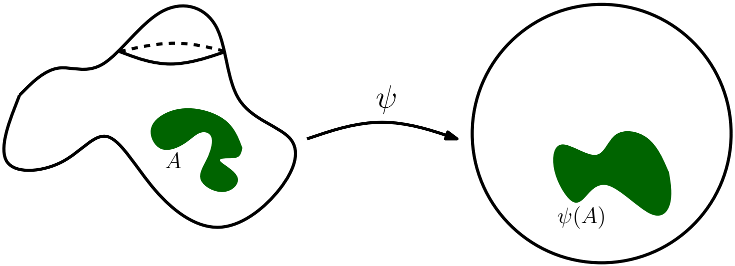

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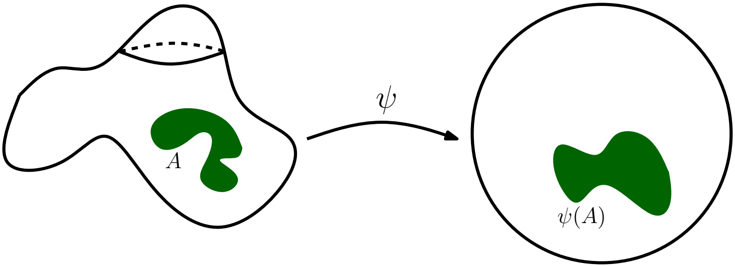

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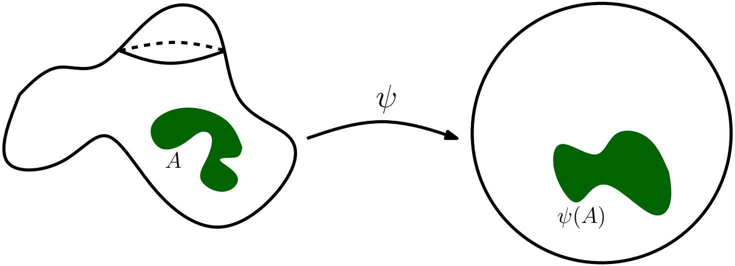

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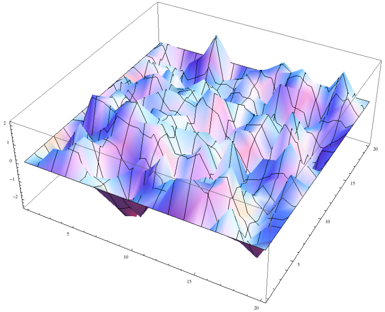
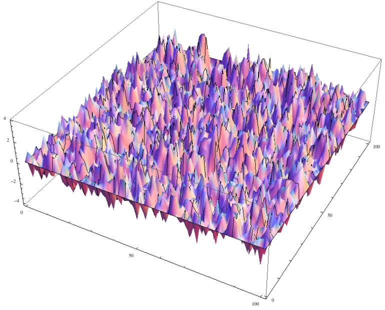
Question: Which measure on ρ? If we want our surface to be a perturbation of a flat metric, natural to choose ρ as the canonical perturbation of a harmonic function.

The Gaussian free field

- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.





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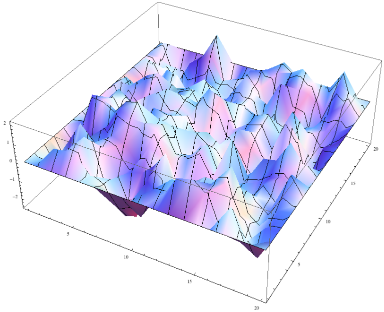
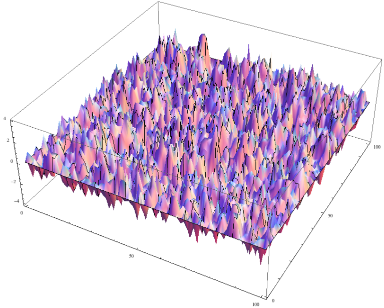
- Measure on functions $h \colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ with density respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$





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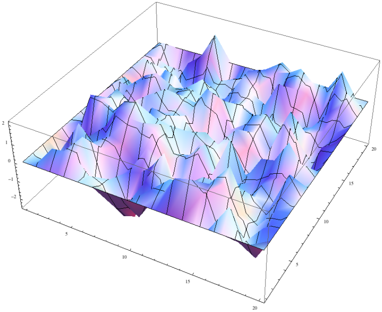

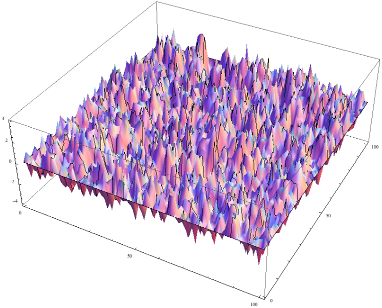

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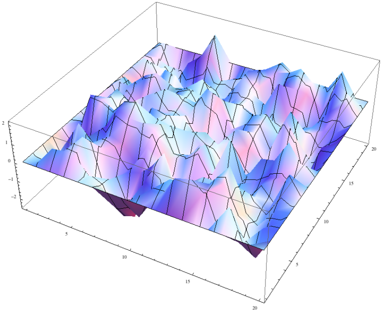

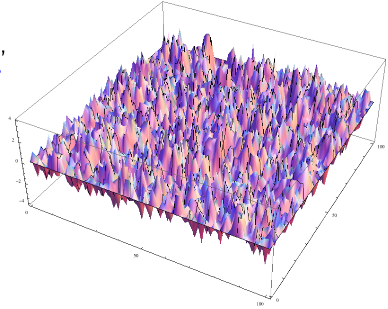

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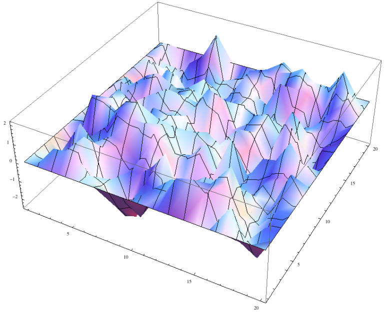

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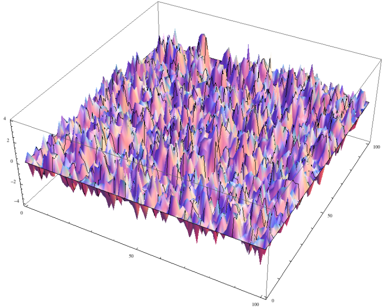

- Continuum GFF not a function — only a generalized function

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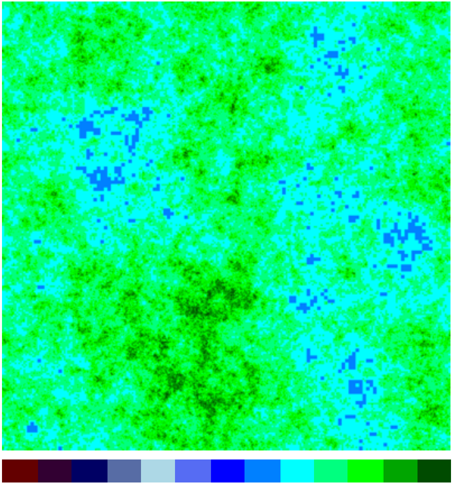

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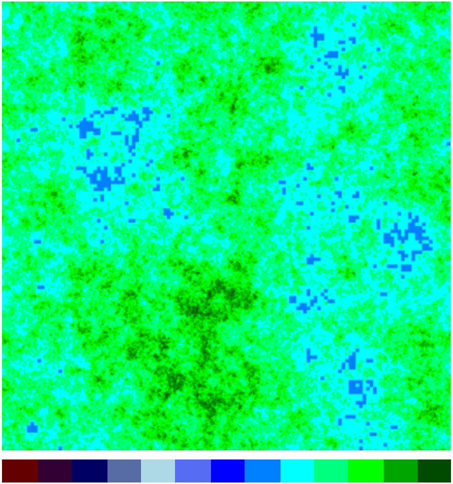

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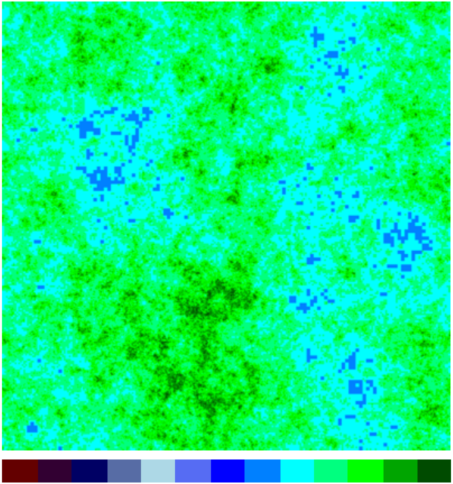

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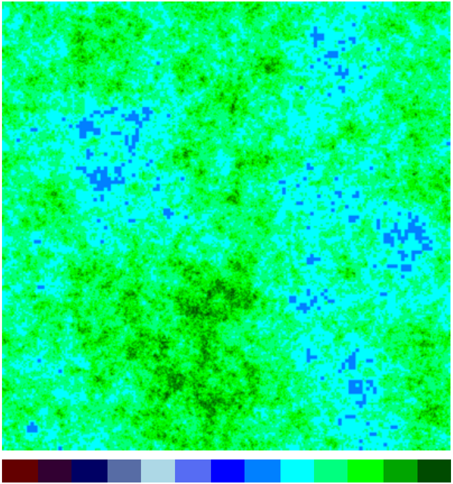

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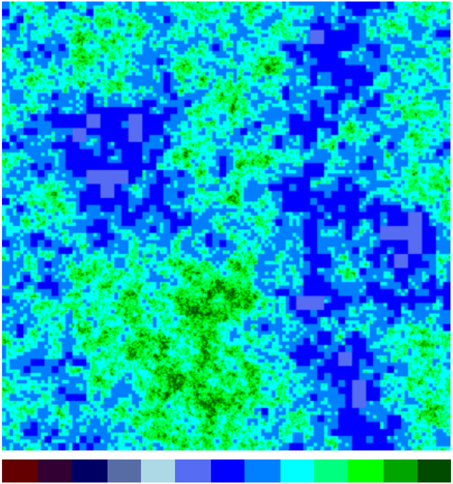

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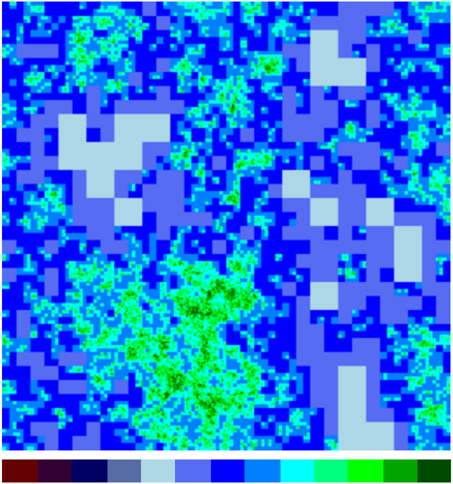

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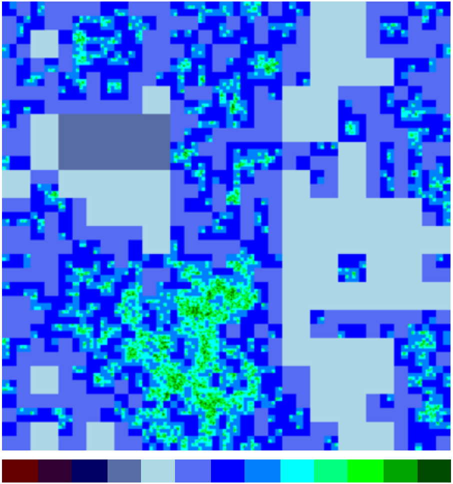

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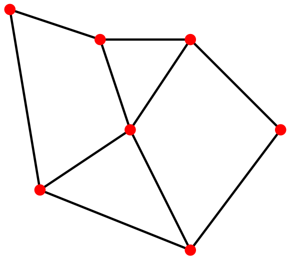

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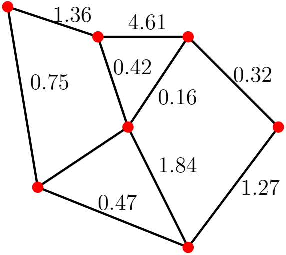

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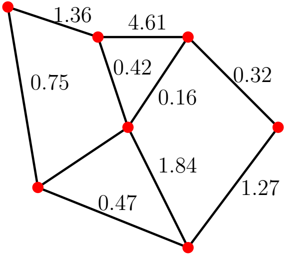

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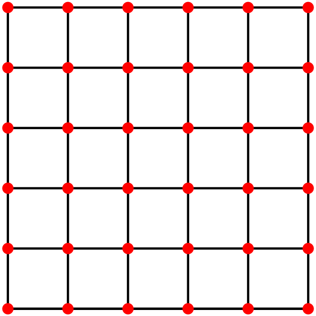

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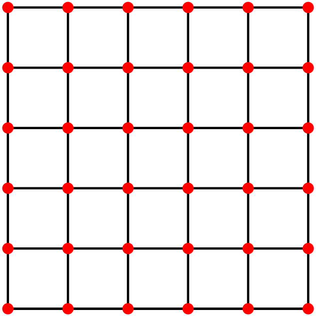

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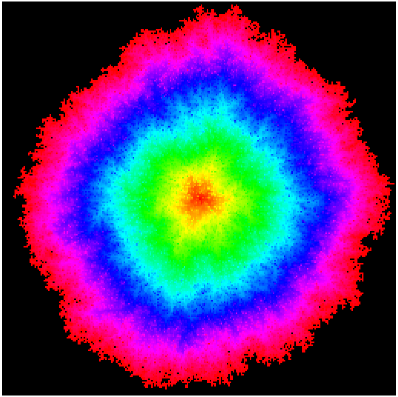

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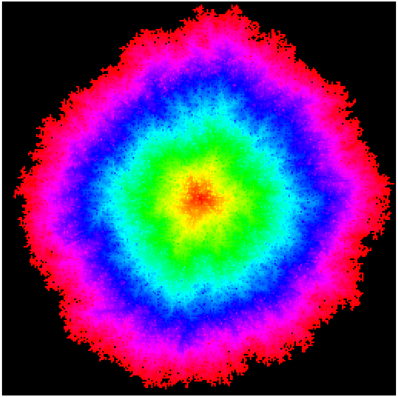

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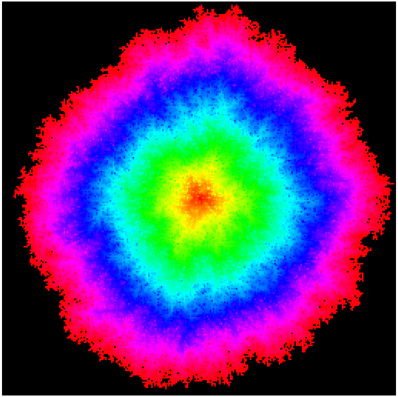

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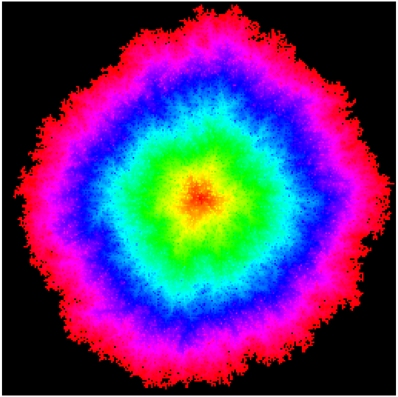

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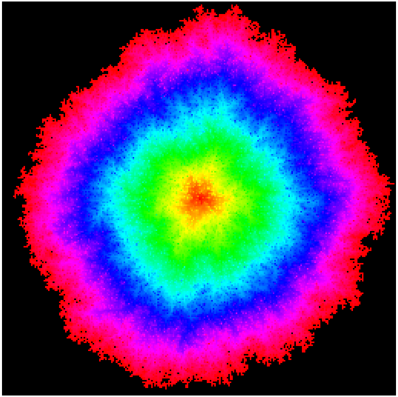
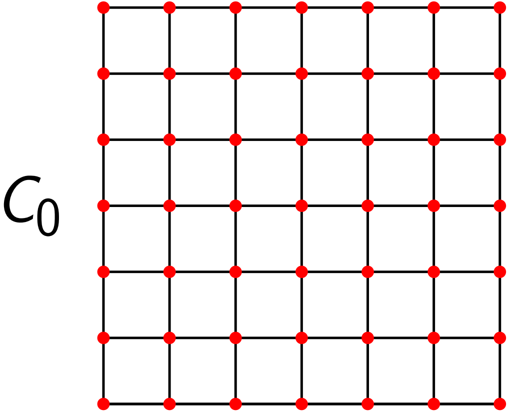


Markovian formulation

Eden exploration



Sample the cluster C_{n+1} from C_n by selecting an edge uniformly at random on ∂C_n, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to η power (η-DBM).

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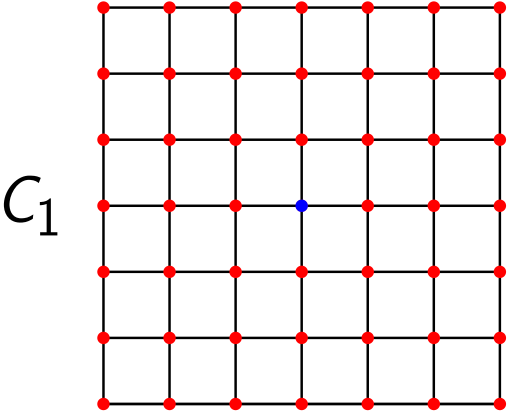

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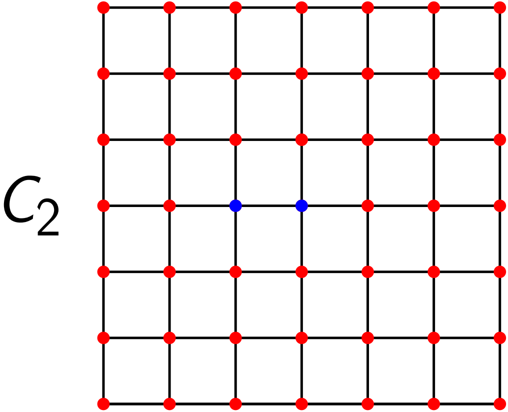

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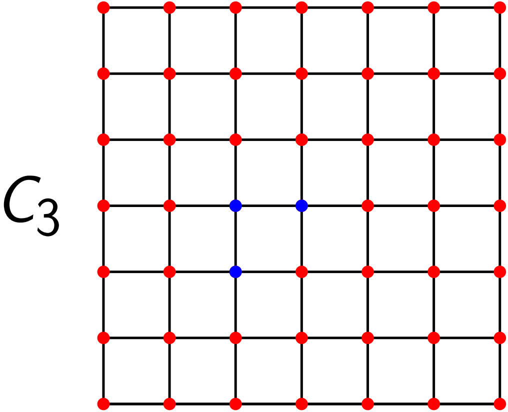

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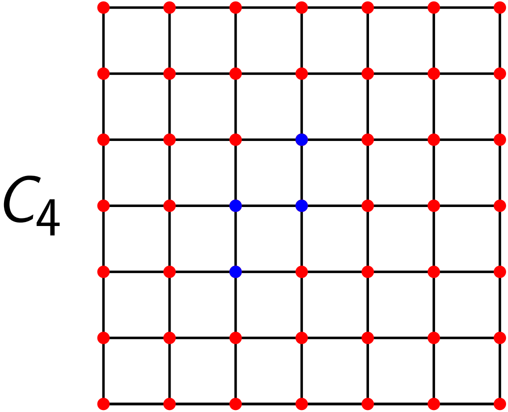

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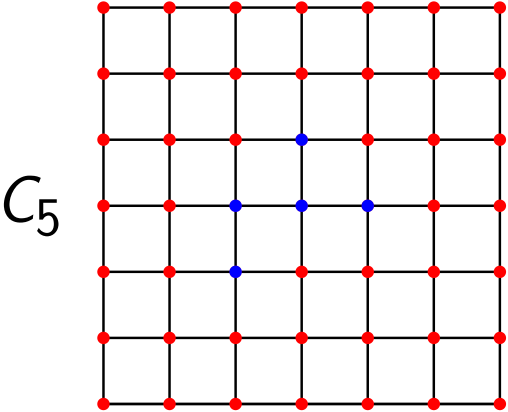

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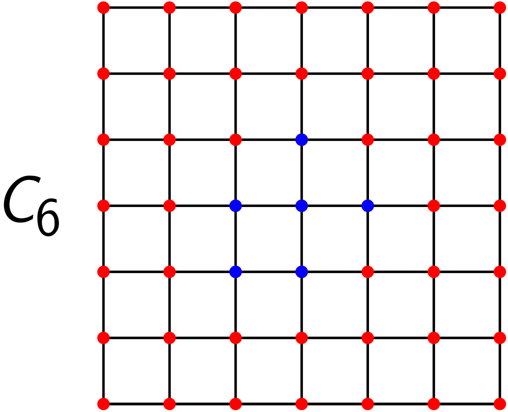

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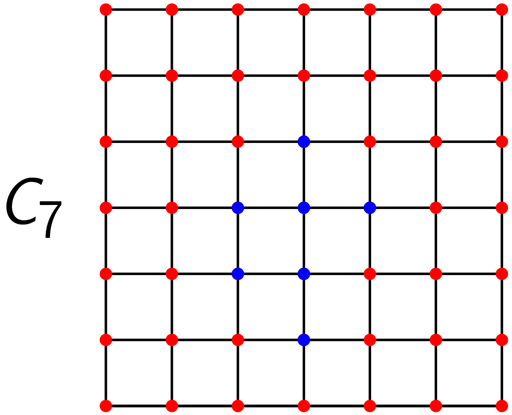

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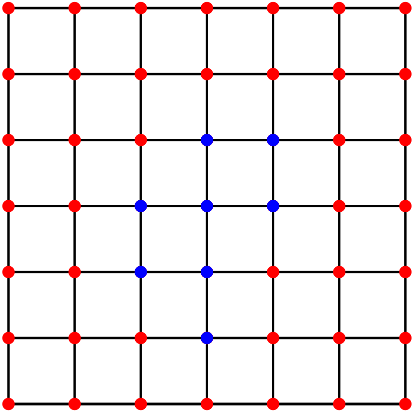
C_8



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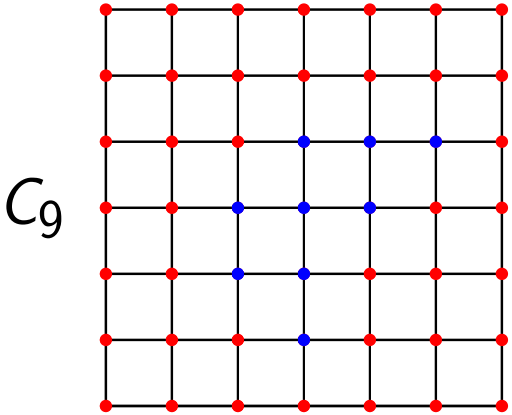

Sample the cluster C_{n+1} from C_n by selecting an edge uniformly at random on ∂C_n, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to η power (η-DBM).

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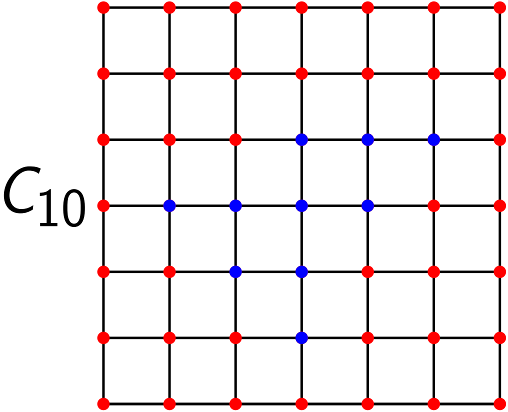

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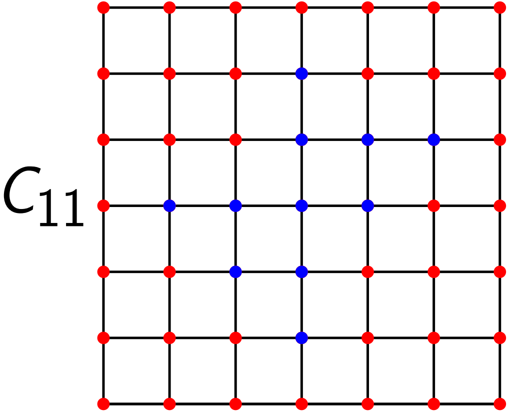

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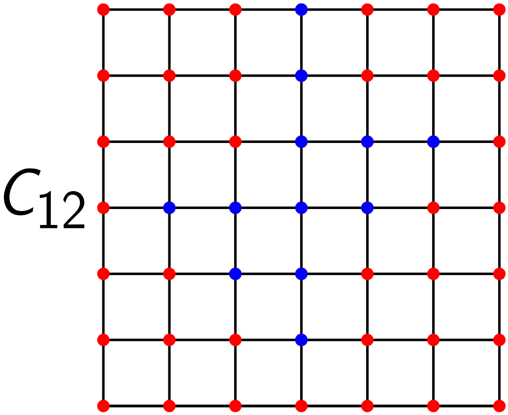

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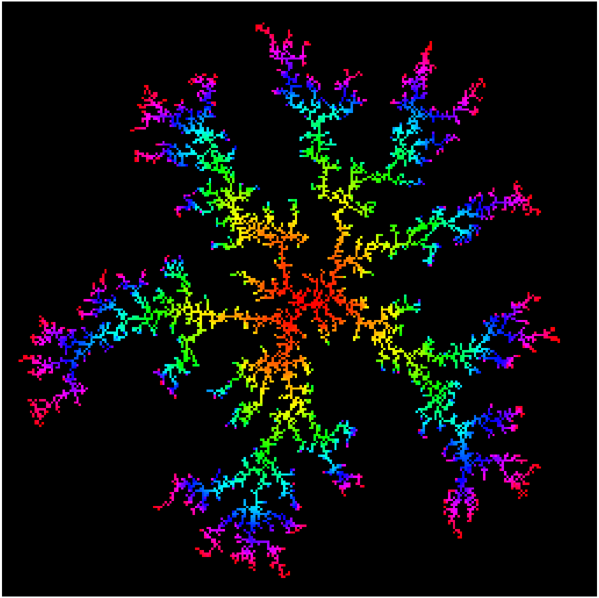
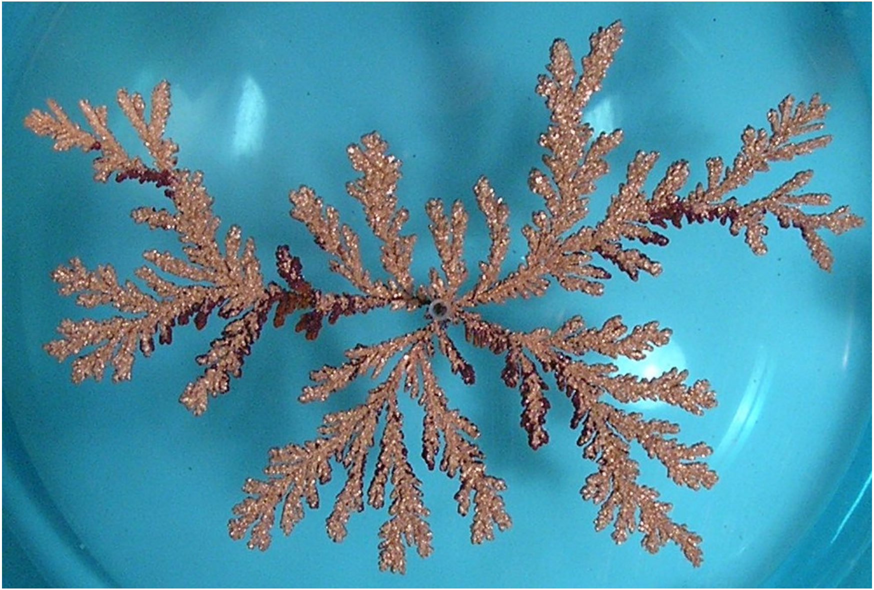


Euclidean Diffusion Limited Aggregation (DLA) introduced by Witten-Sander 1981.



DLA in nature: "A DLA cluster grown from a copper sulfate solution in an electrodeposition cell" (from Wikipedia)

DLA in nature: Magnese oxide patterns on the surface of a rock. (Halsey, Physics Today 2000)

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DLA in art: "High-voltage dielectric breakdown within a block of plexiglas" (from Wikipedia)

Part III:
Basic relationships

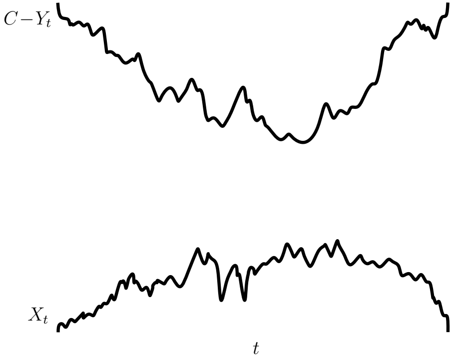
STORY A:

TREE PLUS TREE =
SURFACE PLUS
SELF-HITTING CURVE

independence on both sides

MATING RANDOM TREES

X, Y independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of X and $C - Y$ are disjoint.



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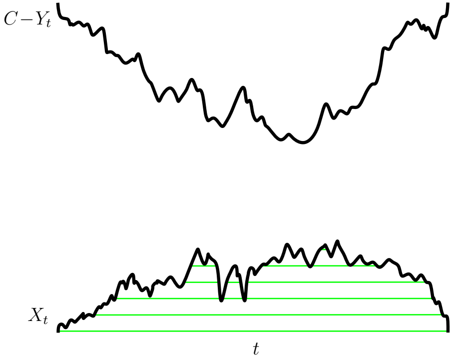

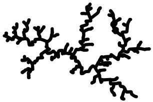

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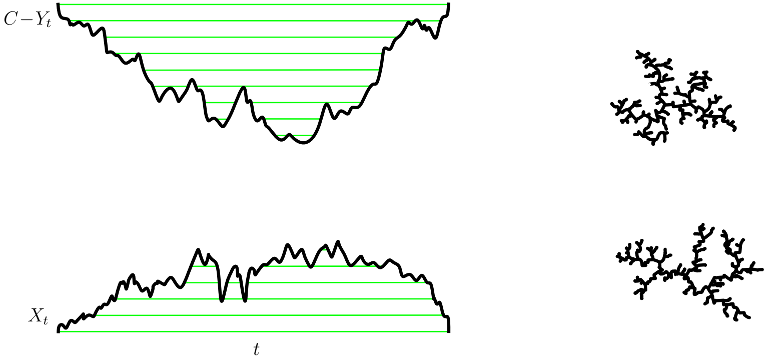

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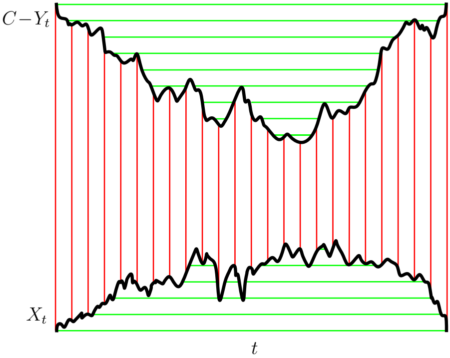

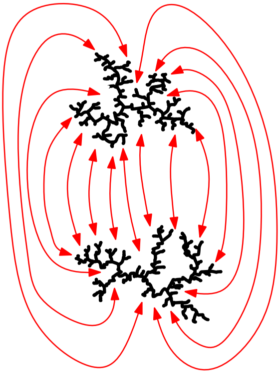

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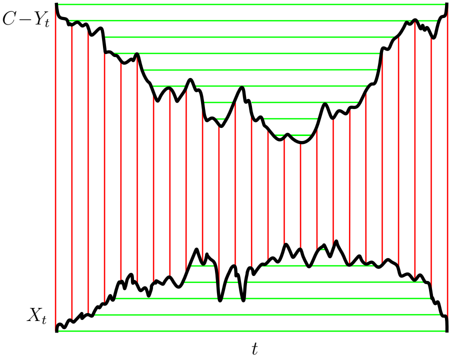

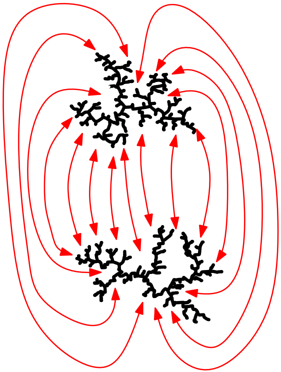

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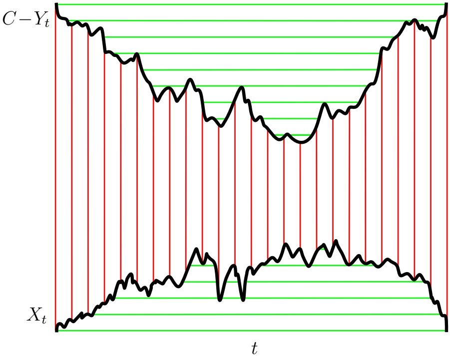

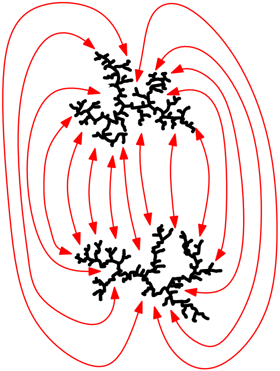

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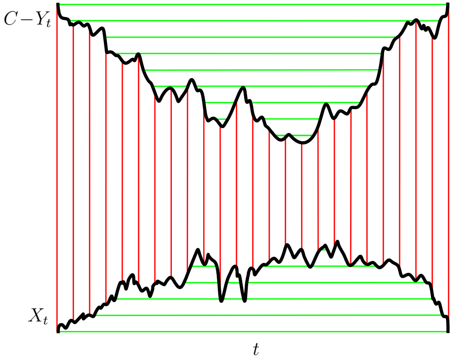

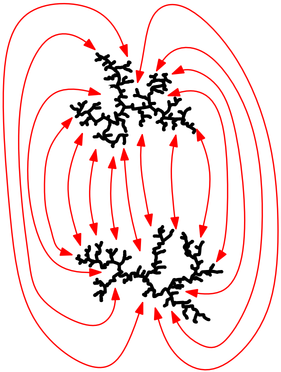

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Surface is topologically a sphere by Moore's theorem

Theorem (Moore 1925)

Let $\cong$ be any topologically closed equivalence relation on the sphere $\mathbf{S}^2$. Assume that each equivalence class is connected and not equal to all of $\mathbf{S}^2$. Then the quotient space $\mathbf{S}^2/\cong$ is homeomorphic to $\mathbf{S}^2$ if and only if no equivalence class separates the sphere into two or more connected components.

- An equivalence relation is topologically closed iff for any two sequences (x_n) and (y_n) with
  - $x_n \cong y_n$ for all n
  - $x_n \to x$ and $y_n \to y$
- we have that $x \cong y$.

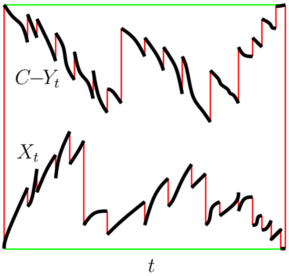
STORY B:

SURFACE TREE PLUS SURFACE TREE = SURFACE PLUS SELF-HITTING CURVE

independence on both sides

Gluing independent Lévy trees

Can view $\mathrm{SLE}_{\kappa'}$ process, $\kappa' \in (4,8)$ as a gluing of two $\frac{\kappa'}{4}$-stable Lévy trees.



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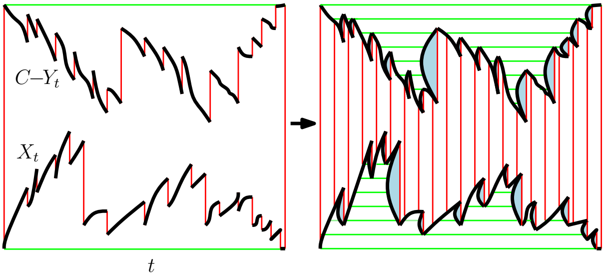

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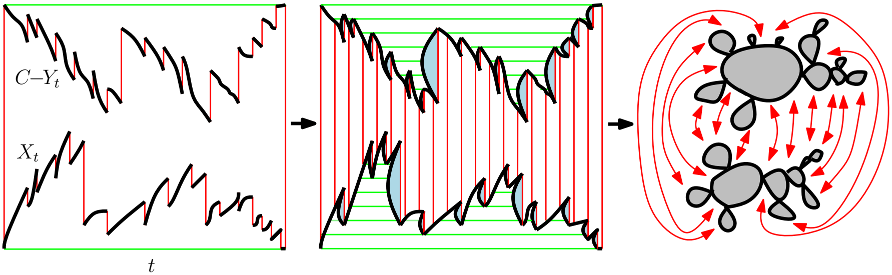

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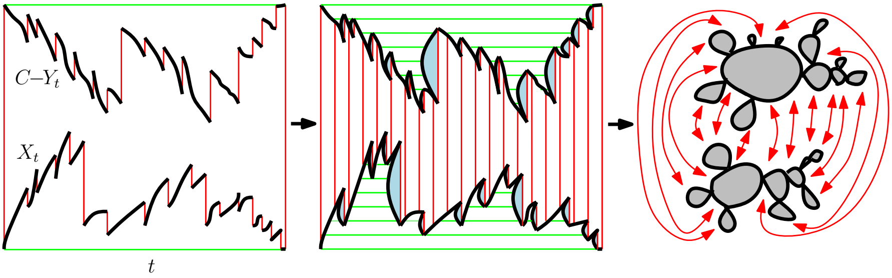

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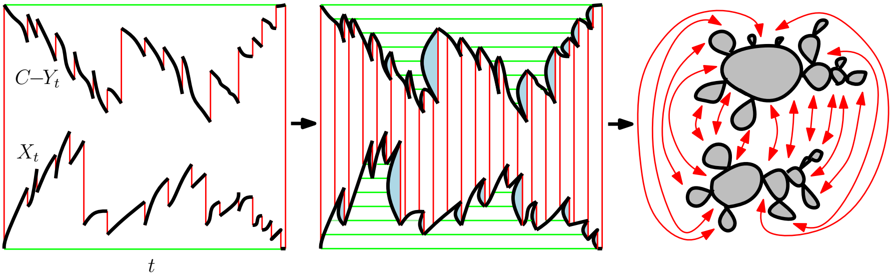

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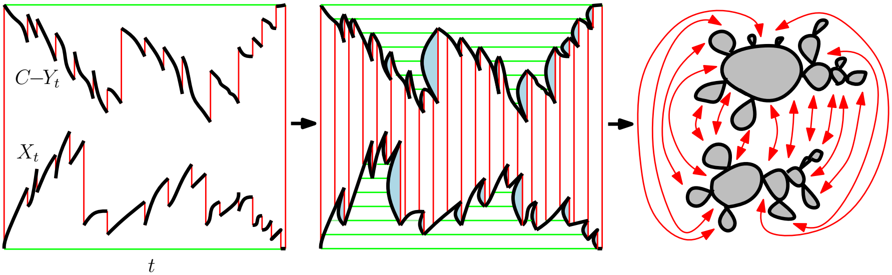

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- Scaling limit of "exploration path" on random planar map should be SLE_6 on a $\sqrt{8/3}$-LQG. Using welding machinery, we can understand well the "bubbles" cut out by such an exploration process. We can understand conditional law of unexplored region given what we have seen.

STORY C:

GROWTH ON SURFACE $=$ "RESHUFFLED" CURVE ON SURFACE

RANDOM GROWTH ON RANDOM SURFACES

- Can we make sense of η-DBM on a γ-LQG? We have shown how to tile an LQG surface with diadic squares of “about the same size” so we could run a DLA on this set of squares and try to take a fine mesh limit.

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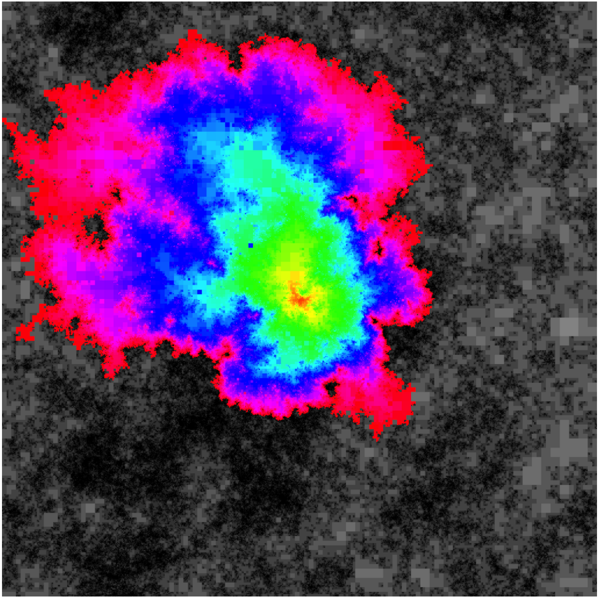
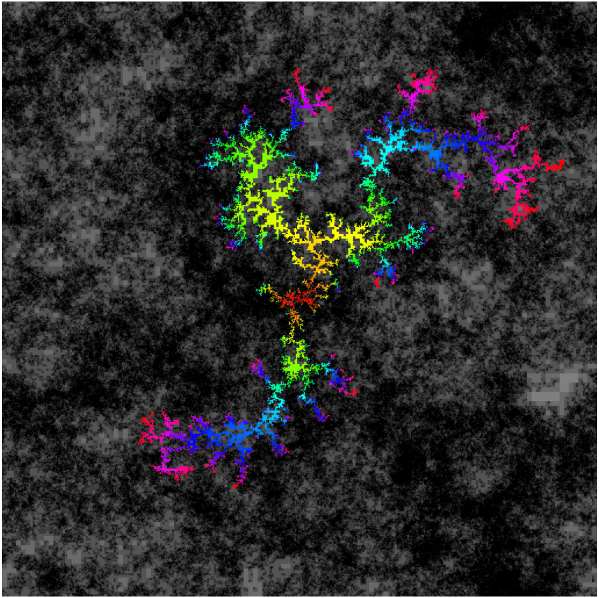
- We will ultimately want to construct a candidate for the scaling limit, which we will call (for reasons explained later) **quantum Loewner evolution:** QLE(γ^2, η).
- But first let's look at some computer generated images (and some animations), starting with an Eden exploration.



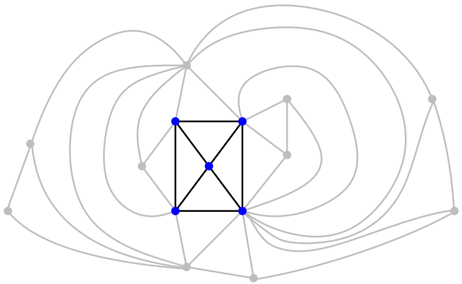
Eden model on $\sqrt{8/3}$-LQG



DLA on a $\sqrt{2}$-LQG

Eden model on planar map

- Random planar map, random vertex x. Perform FPP from x.



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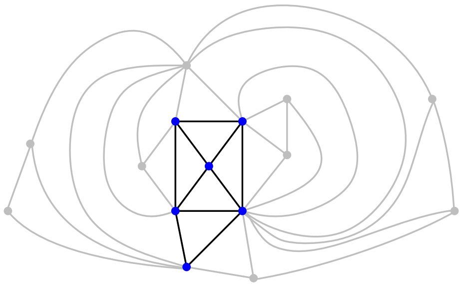

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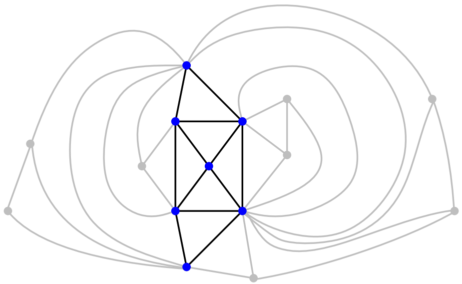

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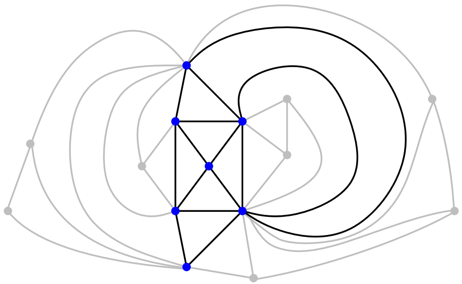

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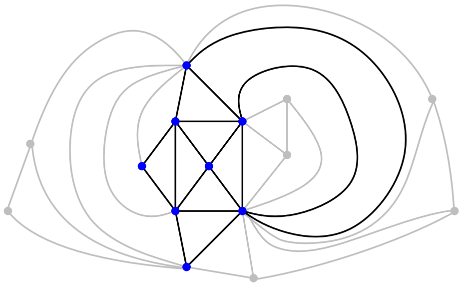

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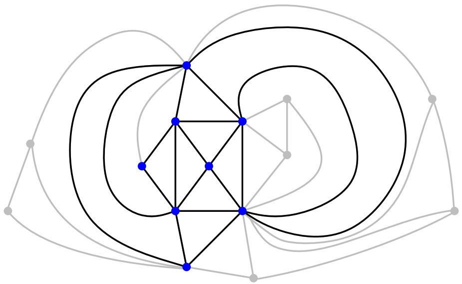

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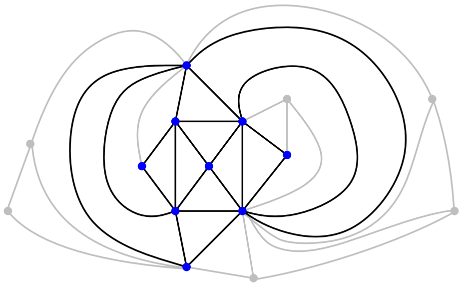

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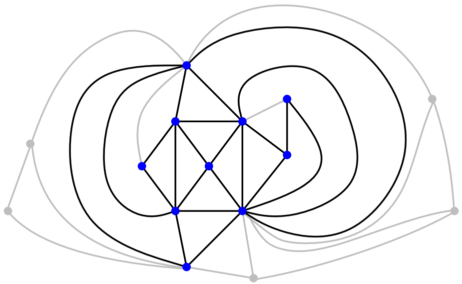

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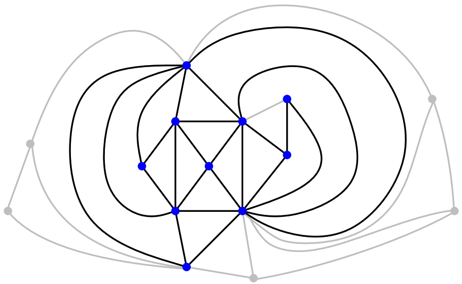

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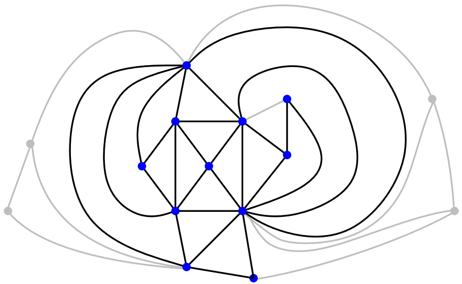

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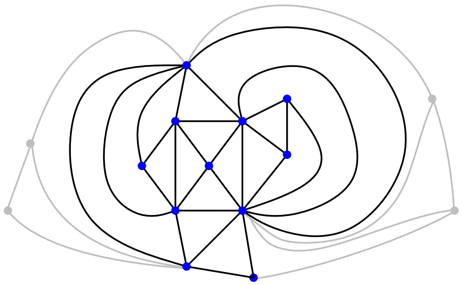

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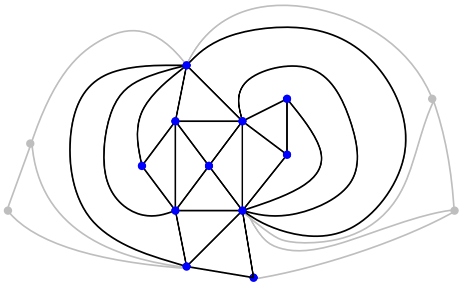

Important observations:

- Conditional law of map given ball at time n only depends on the boundary lengths of the outside components.

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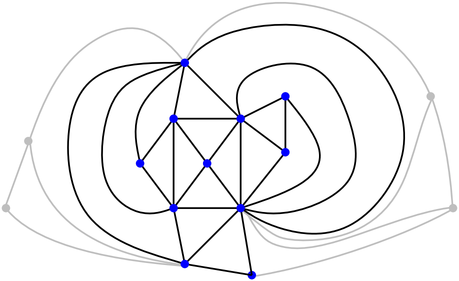

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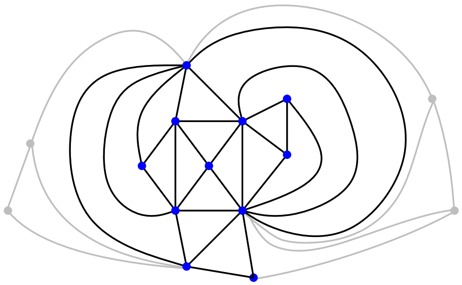

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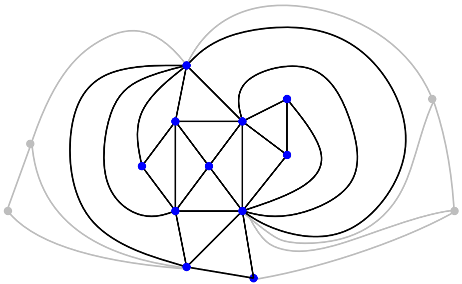

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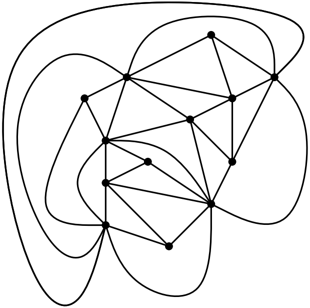
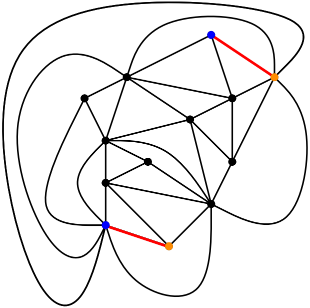
Belief: Isotropic enough so that at large scales this is close to a ball in the graph metric

Continuum limit ansatz



- Sample a random planar map

Continuum limit ansatz



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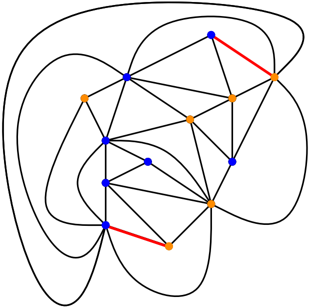
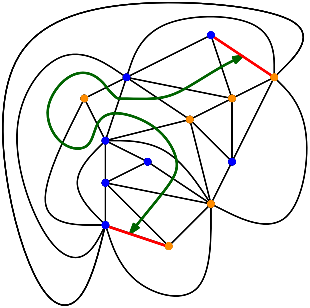


- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability 1/2

Continuum limit ansatz



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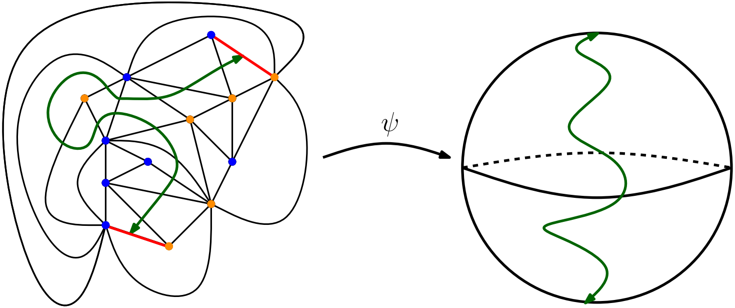

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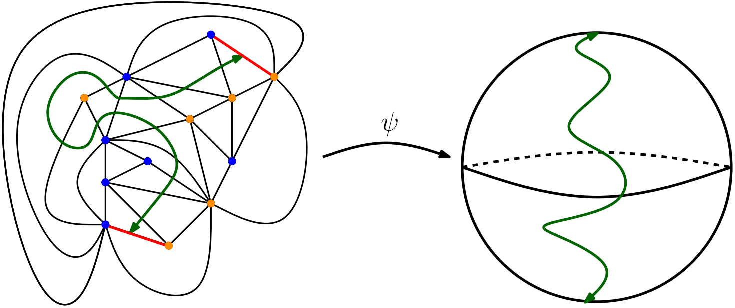

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Ansatz Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent SLE_6.

Continuum analog of first passage percolation on LQG

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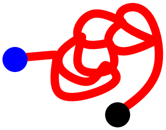

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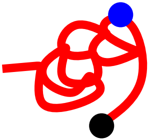

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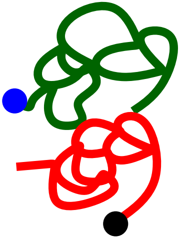

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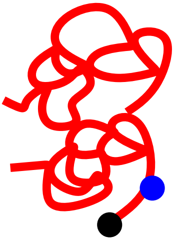

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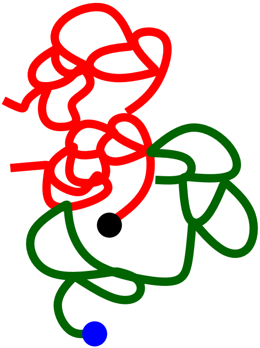

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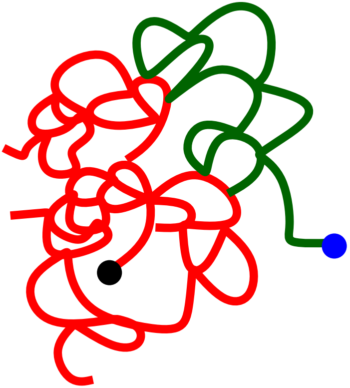

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$\mathrm{QLE}(8/3, 0)$ is SLE_6 with **tip re-randomization**. It can be understood as a "reshuffling" of the exploration procedure associated to the peanosphere.

What is $\mathrm{QLE}(\gamma^2, \eta)$?

$\mathrm{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\mathrm{QLE}(\gamma^2, \eta)$

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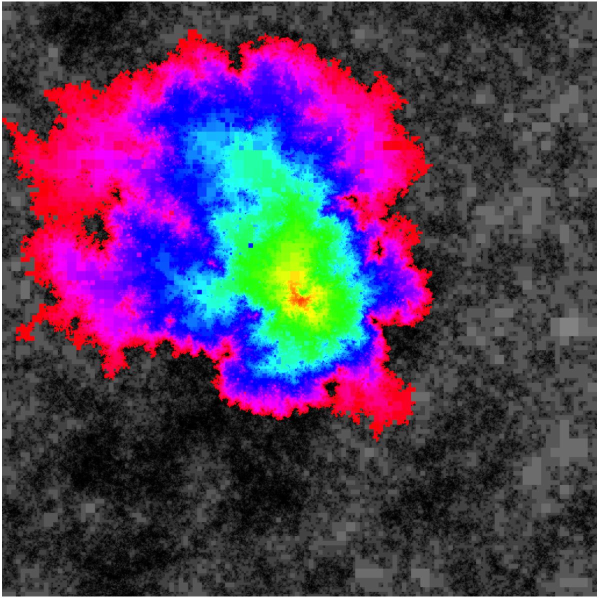
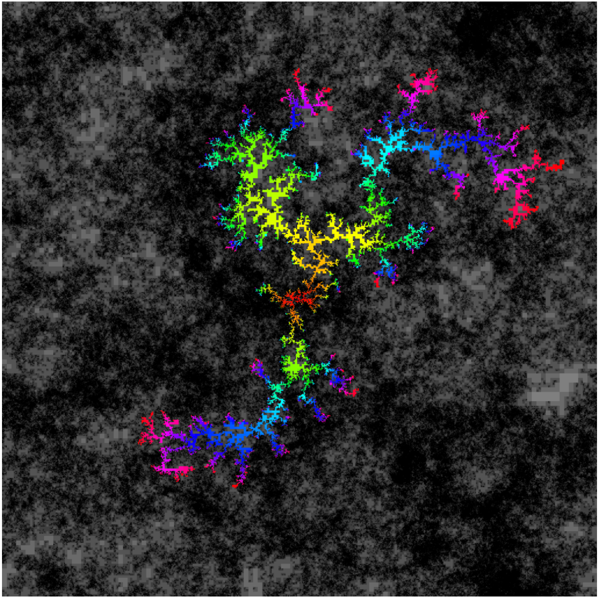
- **First passage percolation:** $\eta = 0$
- **Diffusion limited aggregation:** $\eta = 1$
- **η-dieletric breakdown model:** general values of η

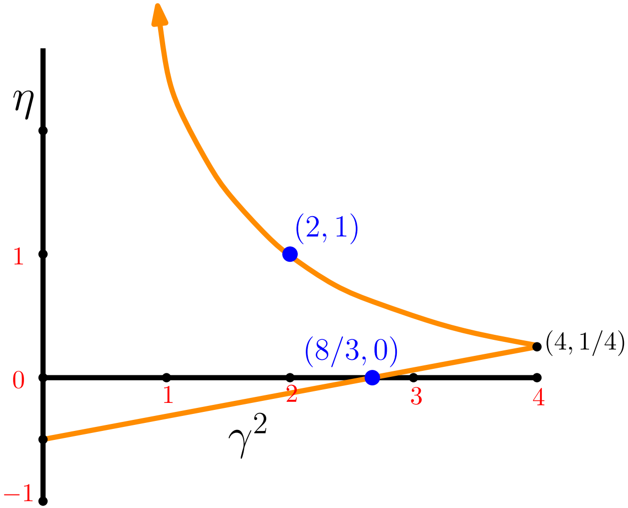


Discrete approximation of $\mathrm{QLE}(8/3, 0)$. Metric ball on a $\sqrt{8/3}$-LQG



Discrete approximation of $\mathrm{QLE}(2,1)$. DLA on a $\sqrt{2}$-LQG

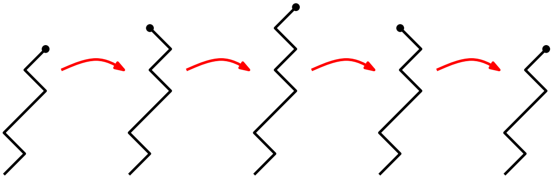
$\mathrm{QLE}(\gamma^2, \eta)$ processes we can construct



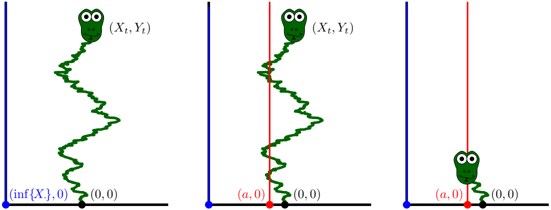
Each of the $\mathrm{QLE}(\gamma^2, \eta)$ processes with (γ^2, η) on the orange curves is built from an SLE_κ process using tip re-randomization.

STORY D:

BROWNIAN MAP $=$ $\sqrt{8/3}$-LIOUVILLE QUANTUM GRAVITY



Dancing snake: a natural random walk on the space of discrete “snakes.”



1. The dancing snake has a scaling limit called the **Brownian snake**.
2. The x and y coordinates of the Brownian snake's head are two functions.
3. Each of these describes a tree (via the same construction we used to make CRT from Brownian motion).
4. Gluing these two trees together gives a random surface called the **Brownian map**.

Some QLE-based results

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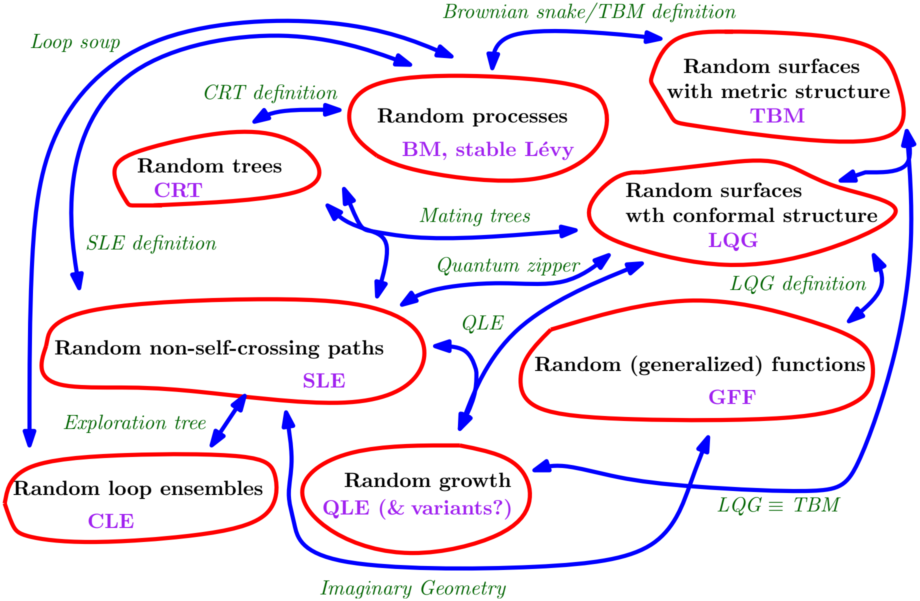
- An understanding of a continuum analog of DLA on a random surface corresponding to $\gamma^2 = 2$.


Brownian snake/TBM definition
Loop soup
Random surfaces
with metric structure
TBM
CRT definition
Random processes
BM, stable Lévy
Random trees
CRT
Random surfaces
wth conformal structure
LQG
Mating trees
SLE definition
Quantum zipper
LQG definition
QLE
Random non-self-crossing paths
SLE
Random (generalized) functions
GFF
Exploration tree
Random loop ensembles
CLE
Random growth
QLE (& variants?)
LQG ≡ TBM
Imaginary Geometry

THANKS!!!