# Conformal blocks in nonrational CFTs with $c \leq 1$

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Based on various joint works with
**Steven M. Flores**, **Alex Karrila** and **Kalle Kytölä**

**Recent developments in Constructive Field Theory, Columbia University**

# Plan

- Belavin-Polyakov-Zamolodchikov (BPZ) PDEs in $2D$ conformal field theory
- "physical solutions": find single-valued ones?
- basis for solution space: "conformal blocks"

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  - Coulomb gas formalism
  - action of the *quantum group* $\mathcal{U}_q(\mathfrak{sl}_2)$ on the solution space
  - can use representation theory to analyze solution space
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  - currently works in the nonrational case
- explicit formulas?
  - integral formulas (Coulomb gas)
  - algebraic formulas when $c = 1$ and $c = -2$

# Introduction:

## BPZ

## PARTIAL DIFFERENTIAL EQUATIONS

# Conformal field theory (CFT)

- consider a $2D$ QFT with "fields" $\phi(z)$
- **impose conformal symmetry:**
  fields $\phi(z)$ carry action of *Virasoro algebra* $\mathfrak{Vir}$
- $\mathfrak{Vir}$: Lie algebra generated by $(L_n)_{n\in\mathbb{N}}$ and central element $C$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12}n(n^2-1)\,\delta_{n+m,0}, \qquad [C, L_n] = 0$$

- central element $C$ acts as a scalar
  = **central charge** $c = \frac{1}{2\kappa}(3\kappa - 8)(6-\kappa)$ (with $\kappa \geq 0$)

[ Initiated by Belavin, Polyakov, Zamolodchikov (1984) ]

# PDE operators from Virasoro generators

- at fixed $z_i$, to each generator $L_{-m}$ one associates a differential operator $\mathcal{L}_{-m}^{(z_i)}$ that acts on *correlations* of the fields:

$$\langle \phi_1(z_1) \cdots (L_{-m}\phi_i(z_i)) \cdots \phi_d(z_d) \rangle = \mathcal{L}_{-m}^{(z_i)} \langle \phi_1(z_1) \cdots \phi_d(z_d) \rangle$$

where $\mathcal{L}_{-m}^{(z_i)} = -\sum_{j \neq i} \left( \frac{1}{(z_j - z_i)^{m-1}} \frac{\partial}{\partial z_j} + \frac{(1-m)\, h_{\phi_j}}{(z_j - z_i)^m} \right)$

- $h_\phi$ are *conformal weights* of the fields $\phi(z)$: transformation rule

$$\langle \phi(z) \cdots \rangle = (\partial f(z))^{h_\phi} \langle \phi(f(z)) \cdots \rangle$$

for conformal maps $f$

**What is the $\mathfrak{Vir}$ -action on a field $\phi(z)$?**

- field $\phi(z)$ generates a $\mathfrak{Vir}$ -module $M_\phi = \mathfrak{Vir}.\phi(z)$
- $M_\phi$ is isomorphic to a quotient of some Verma module $V$:
  $M_\phi \cong V/N$

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# Singular vectors

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- in "degenerate CFTs", $N$ is non-trivial (containing "null fields")
- elements generating $N$ are called **singular vectors:**

$$\mathcal{P}(L_{-m} : m \in \mathbb{N}).\phi(z) \in N$$

where $\mathcal{P}$ is a polynomial in the Virasoro generators

# BPZ partial differential equations

- singular vector $\mathcal{P}(L_{-m} : m \in \mathbb{N}).\phi(z) \in N$ in the quotient $M_\phi \cong V/N$ gives rise to a PDE with $\mathcal{D}^{(z)} = \mathcal{P}(\mathcal{L}_{-m}^{(z)} : m \in \mathbb{N})$

$$\boxed{\mathcal{D}^{(z)} \langle \phi(z) \phi_1(z_1) \cdots \phi_d(z_d) \rangle = 0}$$

where $\mathcal{L}_{-m}^{(z_i)} = -\sum_{j \neq i} \left( \frac{1}{(z_j - z_i)^{m-1}} \frac{\partial}{\partial z_j} + \frac{(1-m)\, h_{\phi_j}}{(z_j - z_i)^m} \right)$

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- Benoit, Saint-Aubin (1988): explicit formulas for singular vectors when conformal weights are of type

$$\frac{s(2s+4-\kappa)}{2\kappa} \qquad \Big( = h_{1,s+1} \text{ or } h_{s+1,1}\Big) \qquad s \geq 0$$

Recall: $c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa)$

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- example: $h_{1,2} = \frac{6-\kappa}{2\kappa}$, field $\phi_{1,2}$ (or $\phi_{2,1}$):

$$\Big( L_{-2} - \frac{3}{2(2h_{1,2}+1)} (L_{-1})^2 \Big)\, \phi_{1,2}(z)$$

(with translation invariance) gives rise to the PDE

$$\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial z^2} + \sum_{i=1}^{d} \left( \frac{2}{z_i - z} \frac{\partial}{\partial z_i} - \frac{2h_{1,2}}{(z_i - z)^2} \right) \right\} \langle \phi_{1,2}(z) \phi_1(z_1) \cdots \phi_d(z_d) \rangle = 0$$

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- **in general**: **homogeneous PDE operators of degree $s+1$**

$$\mathcal{D}^{(z)}_{s+1} = \sum_{n_1 + \ldots + n_j = s} c_{n_1,\ldots,n_j}(s,\kappa) \times \mathcal{L}^{(z)}_{-n_1} \cdots \mathcal{L}^{(z)}_{-n_j}$$

## We seek solutions $F_\varsigma(z;\bar{z})$ to the PDE system

$$\mathcal{D}^{(z_j)}_{s_j+1} F_\varsigma(\boldsymbol{z};\bar{\boldsymbol{z}}) = 0, \qquad \mathcal{D}^{(\bar{z}_j)}_{s_j+1} F_\varsigma(\boldsymbol{z};\bar{\boldsymbol{z}}) = 0, \qquad \text{for all } j = 1,\ldots,d$$

for variables $\boldsymbol{z} = (z_1, z_2, \ldots, z_d)$ and c.c. $\bar{\boldsymbol{z}} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_d)$

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That is, we seek correlation functions

$$\langle \phi_{s_1}(z_1;\bar{z}_1) \cdots \phi_{s_d}(z_d;\bar{z}_d) \rangle = F_\varsigma(\boldsymbol{z};\bar{\boldsymbol{z}})$$

where

- $\varsigma = (s_1, \ldots, s_d)$
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$$\langle \phi_{s_1}(z_1;\bar{z}_1) \cdots \phi_{s_d}(z_d;\bar{z}_d) \rangle = F_\varsigma(\boldsymbol{z};\bar{\boldsymbol{z}}) = \sum_{k,\ell} F^{(k)}(z_1, \ldots, z_d)\, F^{(\ell)}(\bar{z}_1, \ldots, \bar{z}_d)$$

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- **covariance**: for all conformal maps $f$,

$$F_\varsigma(f(\boldsymbol{z}), f(\bar{\boldsymbol{z}})) = \left( \prod_{i=1}^{n} \left( \partial f(z_i)\, \bar{\partial} f(\bar{z}_i) \right)^{-\theta_{s_i}} \right) F_\varsigma(\boldsymbol{z}, \bar{\boldsymbol{z}})$$

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**Theorem** [ Flores, P. (2018+) ]

Suppose $\kappa \in (0,8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa - 8)(6-\kappa) \leq 1$ irrational).

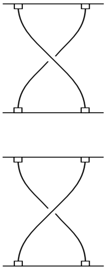
**There exists a unique single-valued, conformally covariant solution $F_\varsigma(z, \bar{z})$ to the Benoit & Saint-Aubin PDEs.**

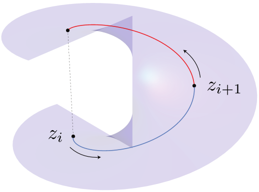
(Uniqueness holds up to normalization in a natural solution space.)

- correlation function $F_\varsigma$ **single-valued** in
  $\mathcal{W}_d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \,\big|\, z_i \neq z_j \ \text{ for } \ i \neq j\}$
- **braid group** $\mathfrak{Br}_d$ is the fundamental group of $\mathcal{W}_d$

invariance: $\quad \sigma_j.F_\varsigma(z,\bar{z}) \;=\; F_\varsigma(z,\bar{z}) \qquad \forall j = 1, 2, \ldots, d-1$

where $\sigma_j$ are generators of the braid group $\mathfrak{Br}_d$



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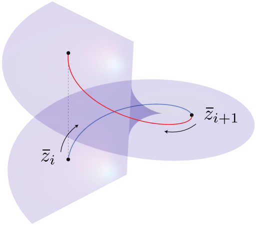


antiholomorphic

**Theorem** [ Flores, P. (2018+) ]

Suppose $\kappa \in (0,8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa-8)(6-\kappa) \leq 1$ irrational).

There exists a unique collection of smooth functions $F_\varsigma(z,\bar{z})$ with $\varsigma = (s_1, \ldots, s_d)$ satisfying $F_{(1)} = 1$ and properties

- **PDE system**:

$$\mathcal{D}^{(z_j)}_{s_j+1} F_\varsigma(z;\bar{z}) = 0, \qquad \mathcal{D}^{(\bar{z}_j)}_{s_j+1} F_\varsigma(z;\bar{z}) = 0, \qquad \forall j = 1, \ldots, d$$

- **covariance**: for all conformal maps $f$,

$$F_\varsigma(f(z), f(\bar{z})) = \left( \prod_{i=1}^{n} \left(\partial f(z_i)\, \bar{\partial} f(\bar{z}_i)\right)^{-\theta_{s_i}} \right) F_\varsigma(z,\bar{z})$$

- **monodromy invariance**: $\sigma_j.F = F$ for all $j = 1, \ldots, d-1$
- **power law growth**: there exist $C > 0$ and $p > 0$ s.t.

$$|\, F_\varsigma \,| \leq C \times \prod_{1 \leq i < j \leq d} \max\left( (z_j - z_i)^p, (z_j - z_i)^{-p} \right)$$

**Corollary**: Asymptotics / OPE:

$$F_{\varsigma}(z;\bar{z}) \underset{z_i,\, z_{i+1}\to \zeta}{\sim} \sum_{s} C^{s}_{s_i,s_{i+1}} \times (z_{i+1}-z_i)^{-\theta_{s_i}-\theta_{s_{i+1}}+\theta_s} \times \text{c.c.}$$

$$\times F_{(s_1,\ldots,s,\ldots,s_d)}(\ldots, z_{i-1}, \zeta, z_{i+2}, \ldots;\ \text{c.c.})$$

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where $s$ belongs to a *finite* index set
and $C^{s}_{r,t}$ are structure constants:

$$C^{s}_{r,t} := \frac{(B^{s}_{r,t})^2 B^{0}_{s,s}}{B^{0}_{r,r} B^{0}_{t,t}} \frac{([s]!)^2 \sqrt{[r+1][s+1][t+1][\frac{r+t-s}{2}]!}}{[\frac{r+s+t}{2}+1]![\frac{s+r-t}{2}]![\frac{t+s-r}{2}]!}$$

$[n]! = [1][2]\cdots[n-1][n]$ and $[n] = \frac{\sin(4\pi n/\kappa)}{\sin(4\pi/\kappa)}$ and

$$B^{s}_{r,t} := \frac{1}{\left(\frac{r+t-s}{2}\right)!} \prod_{i=1}^{\frac{r+t-s}{2}} \frac{\Gamma(1 - \frac{4}{\kappa}(r-i+1))\Gamma(1 - \frac{4}{\kappa}(t-i+1))\Gamma(1 + \frac{4}{\kappa} i)}{\Gamma\left(2 - \frac{4}{\kappa}\left(\frac{r+s+t}{2} - i + 2\right)\right)\Gamma(1 + \frac{4}{\kappa})}$$

**Compare with the work of Dotsenko & Fateev in the 1980s: they calculated the 4-point function and structure constants**

# Construction: conformal blocks

**Ref.** In preparation; see the introduction section 1D in [arXiv:1801.10003] Flores, P. *Standard modules, Jones-Wenzl projectors, and the valenced Temperley-Lieb algebra.*

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$$F(z;\bar{z}) = \sum_{\varrho} \mathcal{U}_{\varrho}(z)\, \mathcal{U}_{\varrho}(\bar{z})$$

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- $\mathcal{U}_{\varrho}$ are *conformal blocks*, which can be written as contour integrals á la Feigin & Fuchs / Dotsenko & Fateev
- Example: when $\varsigma = (1, 1, \ldots, 1)$

$$\mathcal{U}_{\alpha}(\boldsymbol{z}) = \prod_{i<j} (z_i - z_j)^{2/\kappa} \int_{\Gamma_\alpha} \prod_{r} \prod_{j} (w_r - z_j)^{-4/\kappa} \prod_{r<s} (w_r - w_s)^{8/\kappa} \mathrm{d}\boldsymbol{w},$$

where $\Gamma_\alpha$ are certain integration surfaces

See [arXiv:1709.00249] Karrila, Kytölä, P. *Conformal blocks, q-combinatorics, and quantum group symmetry.*

**Theorem** [ Karrila, Kytölä, P. (2017) ] case $\varsigma = (1, 1, \ldots, 1)$

Suppose $\kappa \in (0, 8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa) \leq 1$ irrational).

There exists a unique collection of smooth functions $\mathcal{U}_\alpha(\boldsymbol{x})$

( with $\boldsymbol{x} = (x_1 < \cdots < x_{2N})$ and $\alpha$ planar pair partitions of $2N$ points )

satisfying $\mathcal{U}_\emptyset = 1$ and properties

- **PDE system**: $\forall\ 1 \leq j \leq 2N$,

$$\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left( \frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{6/\kappa - 1}{(x_i - x_j)^2} \right) \right\} \mathcal{U}_\alpha(x_1, \ldots, x_{2N}) = 0$$

- **covariance**: for all (admissible) Möbius maps $\mu \colon \mathbb{H} \to \mathbb{H}$

$$\mathcal{U}(\mu(x_1), \ldots, \mu(x_{2N})) = \prod_j \mu'(x_j)^{\frac{\kappa-6}{2\kappa}} \times \mathcal{U}_\alpha(x_1, \ldots, x_{2N})$$

- specific **asymptotics** (related to fusion)
- **power law growth**: there exist $C > 0$ and $p > 0$ s.t.

$$|\,\mathcal{U}_\alpha\,| \leq C \times \prod_{i<j} \max\left( (x_j - x_i)^p, (x_j - x_i)^{-p} \right)$$

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- **PDE system**
- **covariance**
- **asymptotics**
- **power law growth**

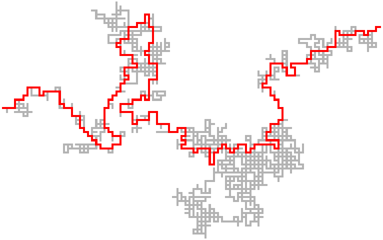
Furthermore:

- $\mathcal{U}_\alpha(x_1, \ldots, x_{2N})$ form **basis** for the solution space
- $\mathcal{U}_\alpha$ give a formula for the unique single-valued (monodromy invariant) correlation function when $\varsigma = (1, 1, \ldots, 1)$
  $F_{(1,1,\ldots,1)}(z; \bar{z}) = \sum_\alpha \mathcal{U}_\alpha(z_1, \ldots, z_{2N})\, \mathcal{U}_\alpha(\bar{z}_1, \ldots, \bar{z}_{2N})$

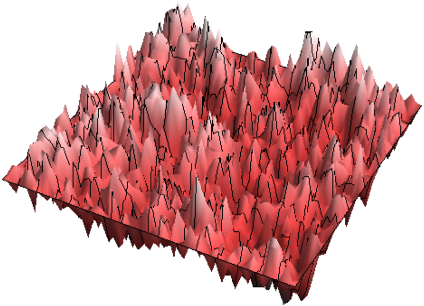
# What's next?

1. Rational examples
2. Ideas for proof

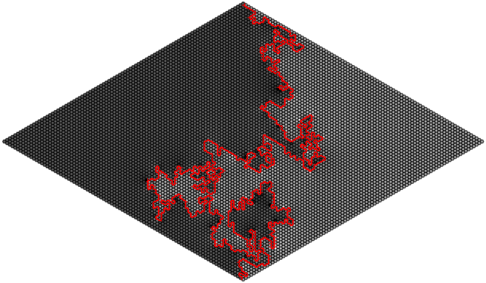
# Loop-erased random walk ($c = -2$)

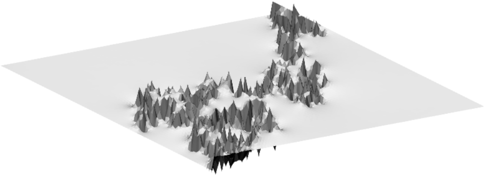


# Gaussian free field ($c = 1$)



# Level line of Gaussian free field: $\mathrm{SLE}_4$ $(c = 1)$



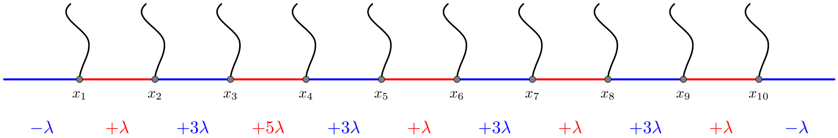


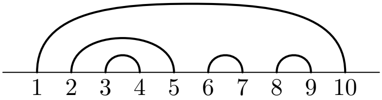
O. Schramm & S. Sheffield

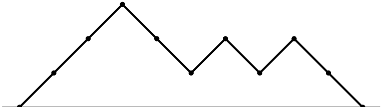
**GFF with boundary data $\pm\lambda$:**

- take boundary values
  $-\lambda$ on the left,
  $+\lambda$ on the right
- **zero level line:**
  $\mathrm{SLE}_\kappa$ **with** $\kappa = 4$
  [Schramm & Sheffield (2003)]

# Conformal blocks for GFF $(c = 1)$



Associate boundary data to "walks"

i.e. planar pairings

$\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$

where $\{a_1, \ldots, b_N\} = \{1, \ldots, 2N\}$,

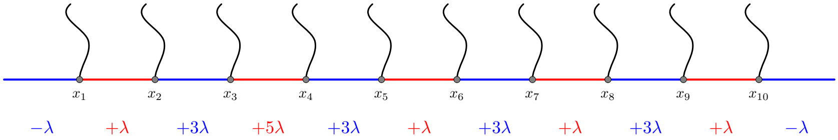
$a_1 < a_2 < \cdots < a_N$,

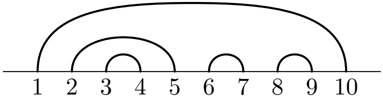
and $a_j < b_j$ for all $j \in \{1, \ldots, N\}$

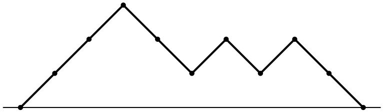
# Conformal blocks for GFF ($c = 1$)

$$\mathcal{U}_\alpha(x_1, \ldots, x_{2N}) = \prod_{1 \le i < j \le 2N} (x_j - x_i)^{\frac{1}{2}\vartheta_\alpha(i,j)},$$

$$\text{where } \vartheta_\alpha(i,j) := \begin{cases} +1 & \text{if } i, j \in \{a_1, a_2, \ldots, a_N\} \text{ or } i, j \in \{b_1, b_2, \ldots, b_N\} \\ -1 & \text{otherwise} \end{cases}$$



Associate boundary data to “walks”

i.e. planar pairings

$\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$

where $\{a_1, \ldots, b_N\} = \{1, \ldots, 2N\}$,

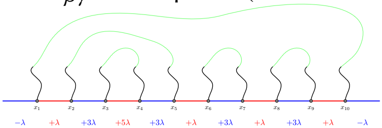
$a_1 < a_2 < \cdots < a_N$,

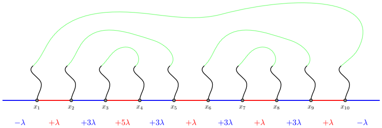
and $a_j < b_j$ for all $j \in \{1, \ldots, N\}$

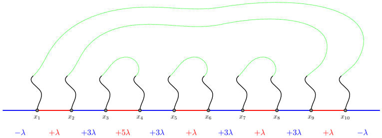
Level lines form some connectivity and, for any $\beta$

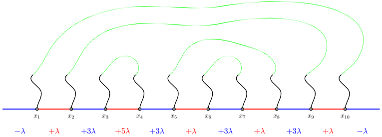
$$\mathbb{P}[\text{connectivity} = \beta] = \frac{\sum_\gamma c_{\beta\gamma}\,\mathcal{U}_\gamma(x_1, \ldots, x_{2N})}{\mathcal{U}_\alpha(x_1, \ldots, x_{2N})}$$

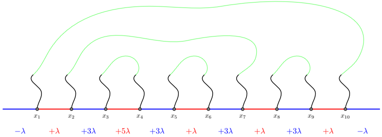
where the coefficients $c_{\beta\gamma}$ are explicit (combinatorial)



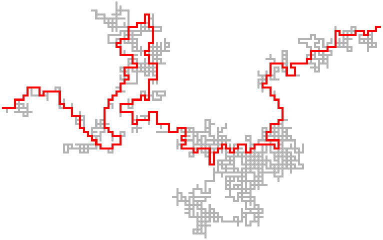



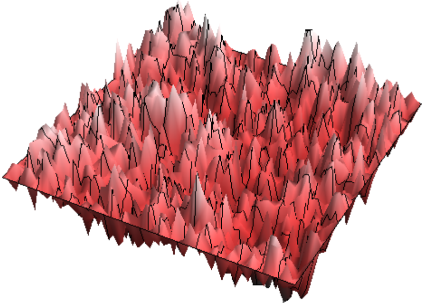




**[arXiv:1703.00898]** P., Wu, *Global and local multiple SLEs for $\kappa \leq 4$ and connection probabilities for level lines of GFF.*

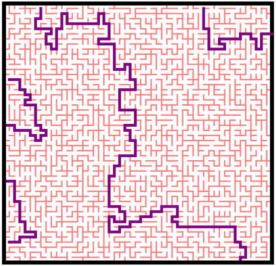
## Loop-erased random walk ($c = -2$)



## Gaussian free field ($c = 1$)



Realize walks as branches in a wired uniform spanning tree:



Connectivities encoded to planar pairings

$\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$

where $\{a_1, \ldots, b_N\} = \{1, \ldots, 2N\}$,

$a_1 < a_2 < \cdots < a_N$,

and $a_j < b_j$ for all $j \in \{1, \ldots, N\}$

The conformal blocks for $c = -2$ (so $\kappa = 2$) are combinatorial determinants á la Fomin, involving Poisson kernels:

$$\mathcal{U}_\alpha(x_1, \ldots, x_{2N}) = \det\Big((x_{a_i} - x_{b_j})^{-2}\Big)_{i,j=1}^{N}$$

**[arXiv:1702.03261]** Karrila, Kytölä, P., *Boundary correlations in planar LERW and UST.*

# General method: Hidden quantum group symmetry

**[arXiv:1408.1384] / JEMS, to appear**

Kytölä, P., *Conformally covariant boundary correlation functions with a quantum group.*

**Theorem** [ Flores, P. (2018+) ]

Suppose $\kappa \in (0,8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa - 8)(6-\kappa) \leq 1$ irrational).

There exists a unique collection of smooth functions $F_\varsigma(z, \bar{z})$ with $\varsigma = (s_1, \ldots, s_d)$ satisfying $F_{(1)} = 1$ and properties

- **PDE system**:

$$\mathcal{D}^{(z_j)}_{s_j+1} F_\varsigma(z; \bar{z}) = 0, \qquad \mathcal{D}^{(\bar{z}_j)}_{s_j+1} F_\varsigma(z; \bar{z}) = 0, \qquad \forall j = 1, \ldots, d$$

- **covariance**: for all conformal maps $f$,

$$F_\varsigma(f(z), f(\bar{z})) = \left( \prod_{i=1}^{n} \left( \partial f(z_i)\, \bar{\partial} f(\bar{z}_i) \right)^{-\theta_{s_i}} \right) F_\varsigma(z, \bar{z})$$

- **monodromy invariance**: $\sigma_j . F = F$ for all $j = 1, \ldots, d-1$
- **power law growth**: there exist $C > 0$ and $p > 0$ s.t.

$$| F_\varsigma | \leq C \times \prod_{1 \leq i < j \leq d} \max\left( (z_j - z_i)^p, (z_j - z_i)^{-p} \right)$$

# How to find solutions of PDEs

$$\mathcal{D}^{(z_j)}_{s_j+1} F = 0 \qquad \text{for all } j = 1, \ldots, d$$

???

Write solution in integral form & use Stokes theorem :

$$F(z_1, \ldots, z_d) = \int_\Gamma f(z_1, \ldots, z_d; w_1, \ldots, w_\ell) \, \mathrm{d}w_1 \cdots \mathrm{d}w_\ell$$

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???

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- find $f$ s.t. for all $j$

  $\mathcal{D}^{(z_j)}_{s_j+1} f$ = "total derivative"

  $\Rightarrow$ then $\mathcal{D}^{(z_j)}_{s_j+1} f\, \mathrm{d}\mathbf{w} = \mathrm{d}\eta$ **is exact $\ell$-form**

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- find $f$ s.t. for all $j$

  $$\mathcal{D}^{(z_j)}_{s_j+1} f \;=\; \text{“total derivative”}$$

  $\Rightarrow$ then $\mathcal{D}^{(z_j)}_{s_j+1} f\,\mathrm{d}\mathbf{w} = \mathrm{d}\eta$ **is exact $\ell$-form**

- find **closed $\ell$-surface** $\Gamma$: $\quad \partial\Gamma = \emptyset$

  $$\Rightarrow \text{ then } \mathcal{D}^{(z_j)}_{s_j+1} F = \int_\Gamma \mathcal{D}^{(z_j)}_{s_j+1} f\,\mathrm{d}\mathbf{w} = \int_\Gamma \mathrm{d}\eta = \int_{\partial\Gamma}\eta = \int_\emptyset \eta = 0$$

# How to find solutions of PDEs

$$\mathcal{D}^{(z_j)}_{s_j+1} \, F = 0 \qquad \text{for all } \; j = 1, \ldots, d$$

???

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**Task: find $f$ and $\Gamma$ such that:**

- $\mathcal{D}^{(z_j)}_{s_j+1} f \;=\;$ "total derivative"
- $\partial\Gamma = \emptyset \qquad \Rightarrow \qquad \mathcal{D}^{(z_j)}_{s_j+1} F = 0$

**1. How to find $f$?** [Feigin-Fuchs / Dotsenko-Fateev '84, "Coulomb gas"]:

$$f \;=\; \prod_{i<j} (z_j - z_i)^{2\alpha_i\alpha_j} \prod_{i,r} (w_r - z_i)^{2\alpha_-\alpha_i} \prod_{r<s} (w_s - w_r)^{2\alpha_-\alpha_-}$$

$$\alpha_i \;=\; \frac{s_i}{\sqrt{\kappa}} \;\text{ and }\; \alpha_- \;=\; -\frac{2}{\sqrt{\kappa}}$$

$$\mathcal{D}^{(z_j)}_{s_j+1} F = 0 \qquad \text{for all } j = 1, \ldots, d$$

???

Write solution in integral form & use Stokes theorem :

$$F(z_1, \ldots, z_d) = \int_\Gamma f(z_1, \ldots, z_d; w_1, \ldots, w_\ell)\, \mathrm{d}w_1 \cdots \mathrm{d}w_\ell$$

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1. **How to find $f$?** [Feigin-Fuchs / Dotsenko-Fateev '84, "Coulomb gas"]
2. **How to choose closed $\Gamma$ s.t. also get**
   - **conformal covariance**
   - **monodromy invariance?**

# How to find solutions of PDEs

$$\mathcal{D}^{(z_j)}_{s_j+1} F = 0 \qquad \text{for all } j = 1, \ldots, d$$

???

Write solution in integral form & use Stokes theorem :

$$F(z_1, \ldots, z_d) = \int_\Gamma f(z_1, \ldots, z_d; w_1, \ldots, w_\ell) \, \mathrm{d}w_1 \cdots \mathrm{d}w_\ell$$

**Task: find $f$ and $\Gamma$ such that:**

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- $\partial\Gamma = \emptyset \qquad \Rightarrow \qquad \mathcal{D}^{(z_j)}_{s_j+1} F = 0$

1. **How to find $f$?** [Feigin-Fuchs / Dotsenko-Fateev '84, "Coulomb gas"]
2. **How to find $\Gamma$?**

**Idea:** Use action of quantum group on integration surfaces $\Gamma$ !

**Theorem** [ Kytölä, P. (2016) & Flores, P. (2018+) ]

Suppose $\kappa \in (0,8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa - 8)(6-\kappa) \le 1$ irrational).

We have an **embedding** of $\mathrm{H}_\varsigma := \{v \in \mathrm{M}_\varsigma \mid E.v = 0,\ K.v = v\}$ onto the solution space

$$\mathcal{F} \colon \mathrm{H}_\varsigma \quad \hookrightarrow \quad \left\{ \text{solutions} \quad \int_\Gamma f(z; \boldsymbol{w}) d\boldsymbol{w} \quad \text{to BSA PDEs} \right\}$$

- $E, F, K$ generators of $\mathcal{U}_q(\mathfrak{sl}_2)$ ( = "quantum" $\mathfrak{sl}_2(\mathbb{C})$ )
- $\mathrm{M}_\varsigma = \mathrm{M}_{(s_1)} \otimes \cdots \otimes \mathrm{M}_{(s_d)}$, where $\mathrm{M}_{(s)}$ is the irreducible type one $\mathcal{U}_q(\mathfrak{sl}_2)$-module of dimension $s+1$

[arXiv:1506.02476] Karrila, Kytölä, P. *Pure partition functions of multiple* SLEs. CMP, 2016

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- when $\varsigma = (1,1,\ldots,1)$, then the map $\mathcal{F}$ is surjective onto

$$\left\{ \begin{array}{l} \text{conformally covariant solutions to} \\ \text{the 2nd order BPZ PDE system } \mathcal{D}_2^{(z_i)} F = 0\ \forall\, i \\ \text{with at most power-law growth} \end{array} \right\}$$

[arXiv:1506.02476] Karrila, Kytölä, P. *Pure partition functions of multiple* SLEs. CMP, 2016

# Quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ = "quantum $\mathfrak{sl}_2(\mathbb{C})$"

- generators $E, F, K, K^{-1}$
- deformation parameter $q = e^{4\pi i/\kappa} \notin e^{\pi i \mathbb{Q}}$
- relations

$$KK^{-1} = 1 = K^{-1}K$$

$$KE = q^2 EK, \qquad KF = q^{-2} FK$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

vectors $v \in \mathrm{M}_{(s_1)} \otimes \cdots \otimes \mathrm{M}_{(s_d)}$ $\longleftrightarrow$ functions $\mathcal{F}[v](z_1, \ldots, z_d)$

**Theorem** [ parts 1-3: Kytölä, P. (JEMS 2018); part 4: "folklore" ]

1. highest weight vectors ( $E.v = 0$ ) $\overset{\mathcal{F}}{\longleftrightarrow}$ solutions of PDEs
2. weight ( $K.v = q^{\lambda} v$ ) $\overset{\mathcal{F}}{\longleftrightarrow}$ conformal transformation properties
3. projections onto subrepresentations $\overset{\mathcal{F}}{\longleftrightarrow}$ asymptotics
4. braiding $\overset{\mathcal{F}}{\longleftrightarrow}$ monodromy

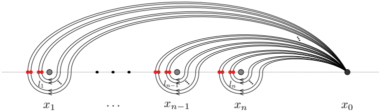
# Hidden quantum group symmetry

hidden quantum group: $\mathcal{U}_q(\mathfrak{sl}_2)$ ( = "quantum" $\mathfrak{sl}_2(\mathbb{C})$ )

vectors $v \in \mathrm{M}_{(s_1)} \otimes \cdots \otimes \mathrm{M}_{(s_d)}$ $\longleftrightarrow$ functions $\mathcal{F}[v](z_1, \ldots, z_d)$

basis vectors $e_{\ell_1} \otimes \cdots \otimes e_{\ell_d}$ $\overset{\mathcal{F}}{\longleftrightarrow}$ integral functions $\displaystyle\int_{\Gamma_{\ell_1,\ldots,\ell_d}} f\, dw$

$$\mathcal{F}[v](z) = \int_{\Gamma(v)} f(z;w)dw$$




$\Gamma_{\ell_1,\ldots,\ell_n}$ $\nearrow$

hidden quantum group: $\mathcal{U}_q(\mathfrak{sl}_2)$ ( = "quantum" $\mathfrak{sl}_2(\mathbb{C})$ )

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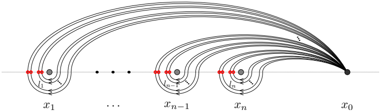


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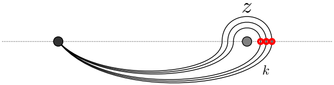
4. braiding $\overset{\mathcal{F}}{\longleftrightarrow}$ monodromy

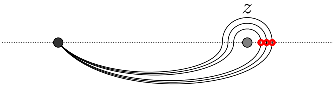
- $\mathrm{M}_{(s)} = \mathrm{span}\{e_0, \ldots, e_s\}$
- $e_0$ is **highest weight vector**: $E.e_0 = 0$ and $K.e_0 = q^s e_0$
- $e_0$ generates $\mathrm{M}_{(s)}$: $e_k = F^k.e_0$
- associate contours to $e_k$:

$e_k$ =




- ***F* adds a contour**:

$F.$

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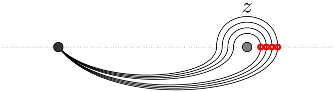
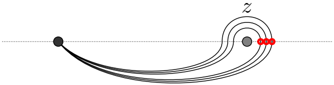
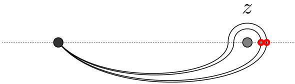



- ***E* removes a contour:**

$E.$

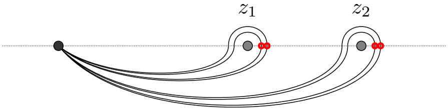



$= \quad [k]_q[d-k]_q\times$



- using coproduct $\Delta : \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$ can **define action on tensor products** of representations:

$$\mathcal{U}_q(\mathfrak{sl}_2) \curvearrowright \mathrm{M}_{(s_1)} \otimes \mathrm{M}_{(s_2)}$$




$$K.(w_1 \otimes w_2) := \Delta(K)(w_1 \otimes w_2) = K.w_1 \otimes K.w_2$$

$$E.(w_1 \otimes w_2) := \Delta(E)(w_1 \otimes w_2) = 1.w_1 \otimes E.w_2 + E.w_1 \otimes K.w_2$$

$$F.(w_1 \otimes w_2) := \Delta(F)(w_1 \otimes w_2) = F.w_1 \otimes 1.w_2 + K^{-1}.w_1 \otimes F.w_2$$

# Calculating monodromy = braiding

vectors $v \in \mathrm{M}_{(s_1)} \otimes \cdots \otimes \mathrm{M}_{(s_d)}$ $\longleftrightarrow$ functions $\mathcal{F}[v](z_1, \ldots, z_d)$

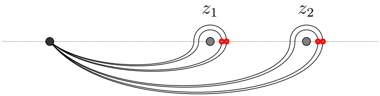
basis vectors $e_{\ell_1} \otimes \cdots \otimes e_{\ell_d}$ $\overset{\mathcal{F}}{\longleftrightarrow}$ integral functions $\int_{\Gamma_{\ell_1,\ldots,\ell_d}} f \, dw$

- consider correlation function

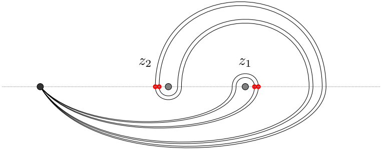
$$F(z_1, \ldots, z_d) = \mathcal{F}[v](\boldsymbol{z}) = \int_{\Gamma} f(\boldsymbol{z}; \boldsymbol{w}) \mathrm{d}\boldsymbol{w}$$

$$f = \prod_{i<j} (z_j - z_i)^{2\alpha_i \alpha_j} \prod_{i,r} (w_r - z_i)^{2\alpha_- \alpha_i} \prod_{r<s} (w_s - w_r)^{2\alpha_- \alpha_-}$$

- monodromy can be calculated by contour deformation method:




$\overset{\sigma}{\longmapsto}$ $e^{\pi \mathrm{i}(\ldots)} \times$



# Conclusion: $\exists\,!$ single-valued solution

**Theorem** [ Flores, P. (2018+) ]

Suppose $\kappa \in (0,8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa - 8)(6-\kappa) \leq 1$ irrational).

**There exists a unique single-valued, conformally covariant solution $F_\varsigma(z,\bar{z})$ to the Benoit & Saint-Aubin PDEs.**

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Suppose $\kappa \in (0,8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa-8)(6-\kappa) \leq 1$ irrational).

**There exists a unique single-valued, conformally covariant solution $F_\varsigma(z,\bar{z})$ to the Benoit & Saint-Aubin PDEs.**

**Proposition** [ Flores, P (2018+) ]

There exists a unique braiding invariant vector $v \in \mathrm{H}_\varsigma \otimes \overline{\mathrm{H}}_\varsigma$.

**Idea**:

- observe that

  { braiding invariant vectors in $\mathrm{H}_\varsigma \otimes \overline{\mathrm{H}}_\varsigma$ } $\cong \mathrm{Hom}_{\mathfrak{Br}_n}(\mathrm{H}_\varsigma, \mathrm{H}_\varsigma)$
- use representation theory to prove that $\dim \mathrm{Hom}_{\mathfrak{Br}_n}(\mathrm{H}_\varsigma, \mathrm{H}_\varsigma) = 1$
- use the "Spin chain – Coulomb gas correspondence" **bijection** $\mathcal{F}\colon \mathrm{H}_\varsigma \otimes \overline{\mathrm{H}}_\varsigma \leftrightarrow$ { solution space } to conclude with the theorem