Lattice gauge theory and string duality

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Outline

- Definition of Lattice gauge theory.

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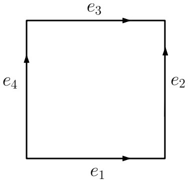
- Playing around with the loop equations and some observations.
- Future directions.

Lattice Gauge theory

- The definition involves a gauge group. We will focus on $SO(N)$.
- Take a finite box $B_n = [-n, n]^d$. Consider a matrix ensemble $\mathcal{Q} = \{Q(e)\}$ indexed by the positively oriented edges $e \in B_n$.
- $Q(e) \in SO(N)$ for all $e \in B_n$, and $Q(x, y) = Q^{-1}(y, x)$
- The main building blocks are the two dimensional four sided loops.



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- if $\beta = 0$ then we have the Haar measure. If β is large and positive then all the Q_p's are forced to be close to the identity matrix.
- Everything is finite and well defined. $\langle \cdot \rangle$ will be used to denote the expectation.

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- Sum over all $(2n-1)(2n-3)\ldots 1$ pair partitions.
- Using this many Gaussian computations are related to counting surfaces.

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only when $j_a = j_{b+1}$ and $j_{a+1} = j_b$ and zero otherwise.

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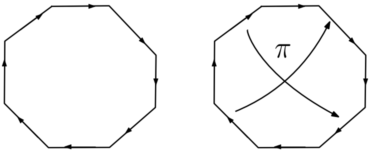

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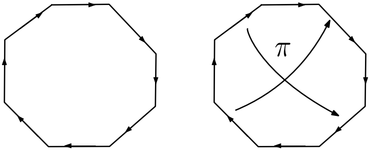

- Multiplicity of a surface is $n^{\text{no of vertices}}$.

Surfaces and partition

- The expression turns out to be of the form $\frac{1}{n^{k/2+1}}\sum_j a_j n^j$ where a_j is the number of surfaces obtained by gluing edges in a compatible way with j vertices

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- A similar thing happens in Lattice Gauge Theory.

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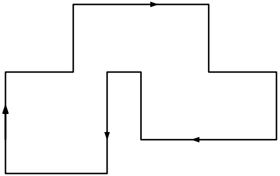
Rigorous versions of Schwinger-Dyson or Master loop equations which is again a clever use of Gaussian integration by parts.

- Later extended to finite unitary setting.

Back to Lattice Gauge and loop variables

- For any lattice loop ℓ, let Q_ℓ be the product of the matrices Q_e along the loop.



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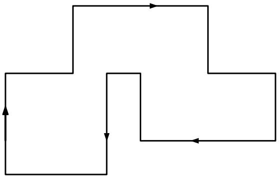

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Assume we have an exhaustion $\Lambda_N \uparrow \mathbb{Z}^d$ and consider the measure on Λ_n with gauge group $SO(N)$.

Key question:

For a loop does $\lim_{N\to\infty} \frac{1}{N}\langle W_\ell \rangle$ exist?

- More generally, for a loop sequence $s = (\ell_1, \ell_2, \ldots, \ell_k)$, does $\lim_{N\to\infty} \frac{1}{N^k}\langle W_s \rangle$ exist? $W_s = \mathrm{Tr}(Q_{\ell_1}) \ldots \mathrm{Tr}(Q_{\ell_k})$.

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Theorem (Gauge string duality (Chatterjee))

There exists $\beta_0 > 0$ such that for $|\beta| < \beta_0$ and for all loop sequence $s = (\ell_1, \ell_2, \ldots, \ell_k)$,

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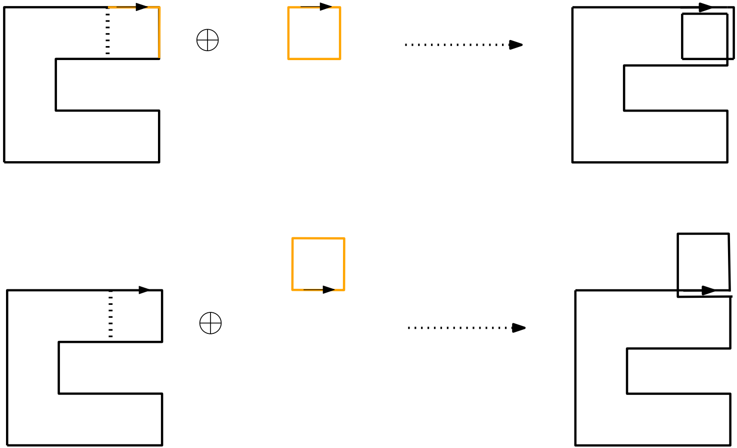
- Trajectories are sequences of strings (loop sequences) obtained by modifying the component loops.

String theory on the lattice

- The integration by parts step makes it necessary to look at a finite sequence of loops in place of a single loop.
- Loop sequences can evolve in time according to certain local modification rules.
- The main objects of interest are trajectories of evolution of such loop sequences ending with the null sequence.
- Each such vanishing trajectory has a weight or action associated to it.

Deformations

- Positive Deformation



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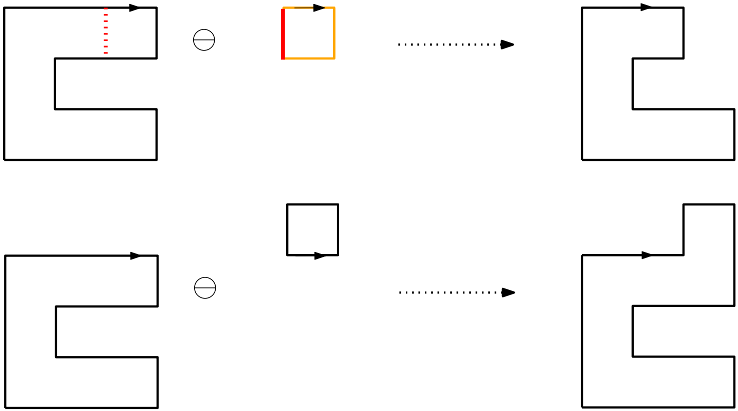

Splitting

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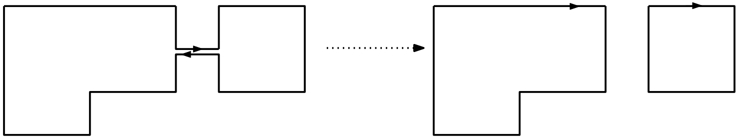

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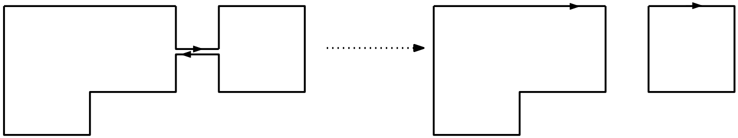

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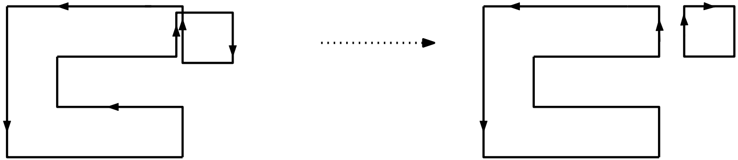

Mergings

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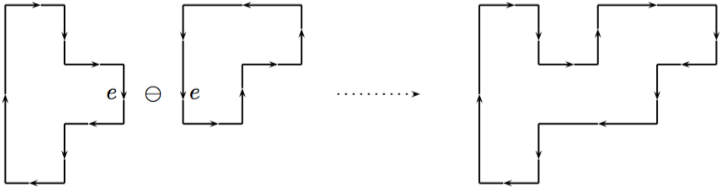

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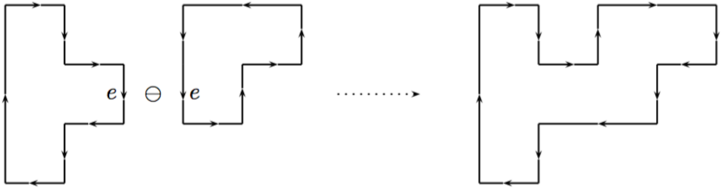

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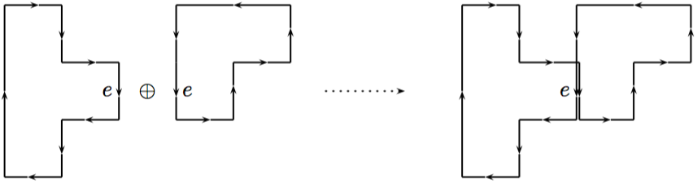

Twisting

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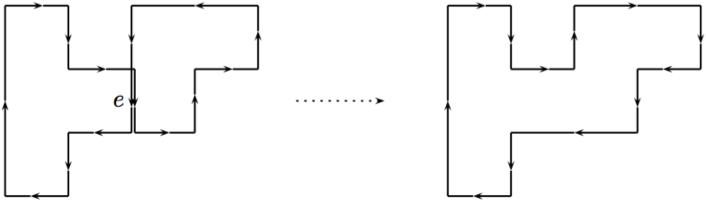

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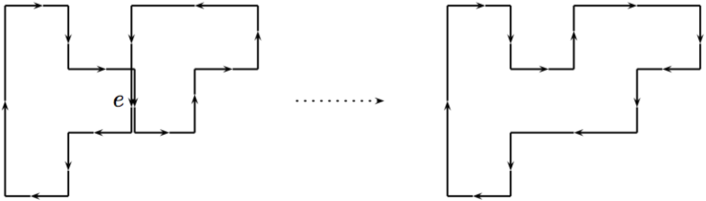

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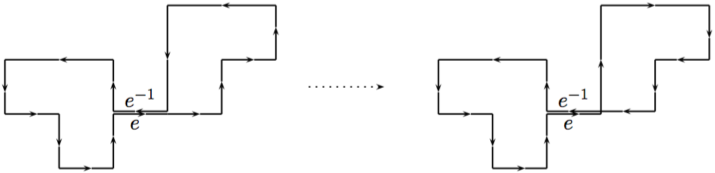

Key identity: Master-loop equation

- Let $\phi(s) := \frac{\langle W_s \rangle}{N^k}$.

The loop equation relates $\phi(s)$ to $\phi(s')$ where s' is obtained from s.

- New feature: True at finite stage before taking limits

$$
\begin{aligned}
(N-1)|s|\phi(s) =& \beta N \sum_{\mathbb{D}^-(s)} \phi(s') - \beta N \sum_{\mathbb{D}^+(s)} \phi(s') + 2N \sum_{\mathbb{S}^-(s)} \phi(s') \\
& - 2N \sum_{\mathbb{S}^+(s)} \phi(s') + 2 \sum_{\mathbb{T}^-(s)} \phi(s') - 2 \sum_{\mathbb{T}^+(s)} \phi(s') \\
& + \frac{2}{N} \sum_{\mathbb{M}^-(s)} \phi(s') - \frac{2}{N} \sum_{\mathbb{M}^+(s)} \phi(s')
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\end{aligned}
$$

- $\mathbb{D}^{\pm}(s) = \{s' :$ obtained by deforming a component loop of $s\}$
- $\mathbb{S}^{\pm}(s) = \{s' :$ obtained by splitting a component loop of $s\}$
- $\mathbb{T}^{\pm}(s) = \{s' :$ obtained by twisting a component loop of $s\}$
- $\mathbb{M}^{\pm}(s) = \{s' :$ obtained by merging component loops of $s\}$

Action/weights of trajectories

- The weight of a trajectory is defined as the following:
- Let m be the total length of all the loops. Then

$$\begin{aligned}
&\frac{\beta}{m} \text{ if the step is a negative deformation}\\
-&\frac{\beta}{m} \text{ if the step is a positive deformation}\\
&\frac{2}{m} \text{ if the step is a negative splitting}\\
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\end{aligned}$$

- The weight of a trajectory is the product of the weights along each step.

Summing up

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- The remaining two operations produce higher genus surfaces and and show up in correction terms i.e.

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- String trajectory: $s = s_1, s_2, \ldots s_m = \varnothing$ where one obtains s_i from s_{i-1} by performing one of the four operations on any one of the loops of the sequence s_{i-1}.
- $a_j(s) =$ weighted(signed) count of trajectories with exactly j deformations. $\lim_{N\to\infty} \frac{\langle W_s\rangle}{N^k} = \sum_{j=0}^{\infty} a_j(s)\beta^j.$
- For β small enough, this is summable.
- The proof relies on bounding the growth rate of the number of trajectories using Catalan type recursions.

Playing around with the loop equation

- One more look at the equation in the 't Hooft limit.

$$\phi(s) := \lim_{N\to\infty} \frac{\langle W_s \rangle}{N^k}$$

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The loop equation has a lot of averaging making it hard to keep track of things.

- Unsymmetrized (localized) version of the loop equation turns out to be more useful.
- Fix an edge e.
- One can write down a version of loop equations where all the operations involve the edge e.
- Note that several copies of e or e^{-1} might occur across s.

Loop recursion in terms of coefficients

- Let ℓ be a loop and e be an edge in ℓ.
- Let A be the position of occurrences of e and B is the position of occurrences of e^{-1}, $C = A \cup B$. Let $m = |C|$.
- Let $\mathcal{P}^+(e)$ denote the set of all positively oriented plaquettes containing the edge e.

$$\begin{aligned} a_k(\ell) &= \frac{2}{m} \sum_{x \in A, y \in B} a_k(\times^1_{x,y}\ell, \times^2_{x,y}\ell) - \frac{1}{m} \sum_{x,y \in A, x \neq y} a_k(\times^1_{x,y}\ell, \times^2_{x,y}\ell) \\ &\quad - \frac{1}{m} \sum_{x,y \in B, x \neq y} a_k(\times^1_{x,y}\ell, \times^2_{x,y}\ell) + \end{aligned}$$

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- The first three terms correspond to splittings and the last two are deformation terms.

Computations for a plaquette

- $a_1 = 1$ since the trajectory of length 1 consists of a single negative deformation.
- Parity considerations force a_2 to be zero.
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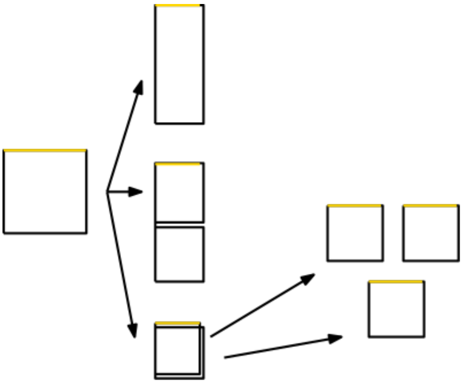

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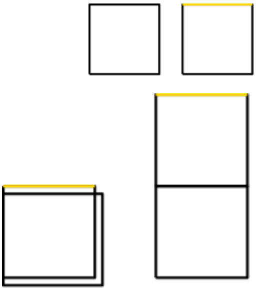

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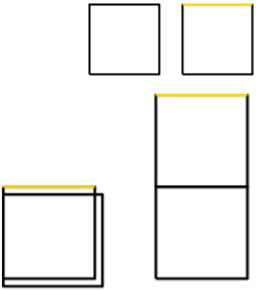

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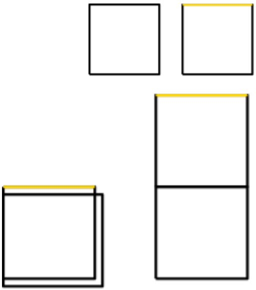

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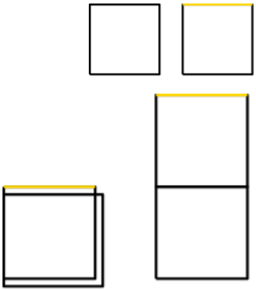

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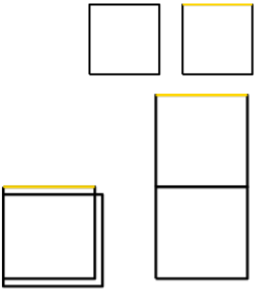

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The above suggests a tree like structure with a spin ± 1 associated to each level to denote whether the orientation of the plaquettes at the j^{th} level is the same or the opposite of that at the $(j-1)^{th}$ level.

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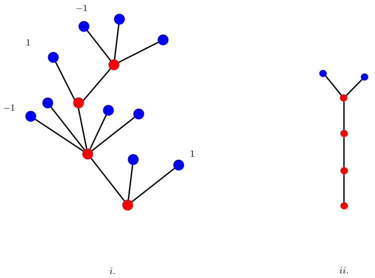

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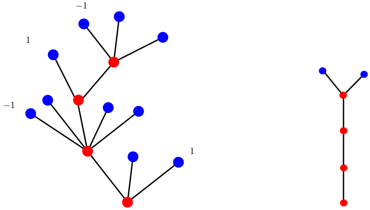

i. ii.

Plaquette variables can be used to determine the free energy.

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- Thus the law of $\tilde{Q}$ is the same as that of Q implying that the marginals are Haar distributed as they are invariant under gauge action.
- One can create complicated gauge functions to simplify many computations.

Connections with Free Probability

- 'Freeness' is the analogue of classical independence in the non-commutative set up.

We would only care about asymptotic freeness of algebras generated by random matrices where the functional is the normalized trace.

- Let S and T be two polynomials.

$$\lim_{N\to\infty} \mathbb{E}(\mathrm{tr}(S(M_N^{(1)})T(M_N^{(2)}))) = \lim_{N\to\infty} \mathbb{E}(\mathrm{tr}(S(M_N^{(1)}))) \lim_{N\to\infty} \mathbb{E}(\mathrm{tr}(T(M_N^{(2)})))$$

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Corollary

The plaquette variables are asymptotically free.

Consequences

- A simple loop of area k can be thought of as a product of k plaquettes.
- Each of them contribute β.
- By freeness the loop variable is β^k.
- For arbitrary loops one gets a polynomial in β i.e. the power series expansion only has finitely many terms.

Freeness of diffusions also comes up in Levy's study of Wilson Loop variables under his construction of two dimensional Yang-Mills theory.

Area Law bounds

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- A gauge theory is said to satisfy area law upper bound if for any loop ℓ of 'area' k, (where the area could be defined in the cell complex terminology)

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One can think of another notion of area being the minimum number of deformations needed to reduce the loop to the null loop.

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- There is a phase transition and large β shows freezing.

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- The linear term does not contribute and the terms beyond the quadratic factors should be negligible.

Higher genus expansions

Recall finite loop equation:

$$(N-1)|s|\phi(s) = \beta N \sum_{\mathbb{D}^-(s)} \phi(s') - \beta N \sum_{\mathbb{D}^+(s)} \phi(s')$$
$$+ 2N \sum_{\mathbb{S}^-(s)} \phi(s') - 2N \sum_{\mathbb{S}^+(s)} \phi(s')$$
$$+ 2 \sum_{\mathbb{T}^-(s)} \phi(s') - 2 \sum_{\mathbb{T}^+(s)} \phi(s')$$
$$+ \frac{2}{N} \sum_{\mathbb{M}^-(s)} \phi(s') - \frac{2}{N} \sum_{\mathbb{M}^+(s)} \phi(s')$$

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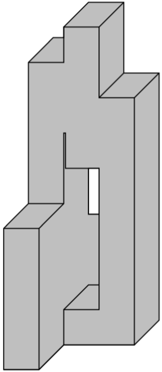
&+ 2N \sum_{\mathbb{S}^-(s)} \phi(s') - 2N \sum_{\mathbb{S}^+(s)} \phi(s') \\
&+ 2 \sum_{\mathbb{T}^-(s)} \phi(s') - 2 \sum_{\mathbb{T}^+(s)} \phi(s') \\
&+ \frac{2}{N} \sum_{\mathbb{M}^-(s)} \phi(s') - \frac{2}{N} \sum_{\mathbb{M}^+(s)} \phi(s')
\end{aligned}
$$



(Chatterjee, Jafarov.)

Counts of higher genus objects form the coefficients of the correction terms $\frac{1}{N}, \frac{1}{N^2} \dots$.

Higher genus expansions

Recall finite loop equation:

$$
\begin{aligned}
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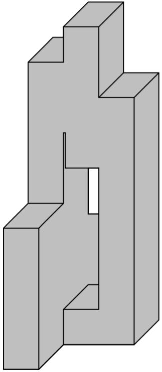

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Counts of higher genus objects form the coefficients of the correction terms $\frac{1}{N}, \frac{1}{N^2} \ldots$. Recall we saw a similar phenomenon in the Gaussian case.

Concluding Remarks and Future Directions

- The free probability arguments special to the planar case however help us observe that a natural non-crossing structure exists in the trajectories in any dimension (the pair partition being formed in a way so that elements of a pair cancel each other)

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- Can one prove something interesting in the Gaussian theory appropriately weighted to approximate Unitary matrices?

THANK YOU!