ALGEBRAIC CURVES
SECOND HALF OF THE SEMESTER

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There are twelve lectures left in the semester. Here is what I plan to cover, with references, and tentative dates. I would appreciate feedback as to how much you already know of the material, especially the topology and the advanced calculus.

1. Definition of Riemann Surfaces and Holomorphic Maps by Charts

Tuesday, March 20th.

I will mainly follow Kirwan [4], §5.2. Note that we will skip the Weierstrass $\mathfrak{p}$-function for the time being.

First we define topological surfaces, p. 125, using atlases. Then the definition of a Riemann surface, p. 132, with examples. The key example for us is any affine plane curve with its singular points removed, and any projective plane curve with its singular points removed. I will assign Exercises from p. 138: 1, 2, 3, 4, 5, 6, 16. Then we will cover:

The inverse function theorem, which is proved in the same way as we proved the complex implicit function theorem - Kirwan, Theorem B.1.

A lemma that shows that that one can extract the n-th root of an infinite series with complex coefficients of the form $1 + b_1 z + b_2 z^2 + \dots$.

Kirwan's Exercise 16 mentioned above, which gives a local "normal form" for holomorphic maps.

Finally we will define a *proper* map: any map $f\colon S \to T$ between topological spaces such that the inverse of a compact set is compact.

A more advanced reference with some useful material for beginners at the start of the book is Donaldson [1]. Some of the results above are covered in Donaldson [1].

2. Fundamental Groups and Covering Spaces

Thursday, March 22th.

This lecture is mainly topology. The best reference here is Munkres [6]. In this lecture and the next, we will cover the fundamental group π_1, defined in Chapter 9, and the classification of covering spaces, in Chapter 13. Please let me know how much of this material you know. Today we will cover §51, §52, §53, §54. We will compute the fundamental group of all our model compact surfaces, namely spheres with g handles. We ask what happens if one removes a finite set of points from these surfaces.

Date: March 16, 2012.

3. Riemann Surfaces and Covering Spaces

Tuesday, March 27th.

We start with the topology of covering spaces: Munkres, Chapter 13. We will cover §79, §80, §82. Since all the spaces we will consider are locally homeomorphic to $\mathbb{R}^2$, the hypotheses of Munkres needed to prevent pathologies are satisfied: our spaces are locally path connected and path connected.

Then we connect the algebra of the fundamental group with d-sheeted covering spaces: the key notion is that of a *transitive permutation representation* of a group π: a homomorphism of π to the group $\mathfrak{S}_d$ of permutations of d elements $e_1, \dots, e_d$, such that the stabilizer of e_i is a subgroup of index d. Then we define the *monodromy* of the covering

Next we prove the major result of this lecture:

Theorem 3.1 (Riemann's Existence Theorem)**.** *Let Y be a connected Riemann surface, and Δ a discrete set of point of Y. Given a $d \geq 1$ and a transitive permutation representation $\rho\colon \pi_1(Y \setminus \Delta) \to \mathfrak{S}_d$, there is a connected Riemann surface X and a proper holomorphic map $F\colon X \to Y$ such that ρ is the associated monodromy. Furthermore X and F are unique up to equivalence.*

See Donaldson [1], p. 49.

Finally we use this to compactify algebraic curves, and to desingularize algebraic curves. Donaldson, pp. 50-54.

4. Differential Forms and Integration on Surfaces

Thursday, March 29th.

This is pure calculus, but on real surfaces defined with charts. We define differential one-forms and two-forms, and learn how to integrate them on curves and on pieces of surfaces. The big theorem is Stokes' Theorem, which you may have seen in advanced calculus. The treatment in Donaldson, Chapter 5, is very nice. Miranda [5] is also good, but he does not treat the C^∞ case before the complex case, so he covers the material in lectures 4-6 in a different order. Kirwan only treats the complex analytic case.

Question: have you taken Calculus IV or the equivalent? Have you seen Stokes' Theorem before?

5. de Rham cohomology

Tuesday, April 3rd.

My reference is Donaldson, §5.2, pp 67-73. Most discussions of de Rham cohomology are written for manifolds of all dimensions. His is clean and short, and most importantly, only deals with surfaces.

6. Differential Forms and Integration on Riemann Surfaces

Thursday, April 5th.

The previous two lectures held on any surface. Now we see what happens when the surface has a complex structure, and so is a Riemann surface. This is covered in Kirwan, §6.1, in Donaldson §5.3, in Miranda, Chapter IV.

7. Elliptic Integrals, Theta Functions and the Weierstrass $\mathfrak{p}$-function

Tuesday, April 10th.

This is partly in Kirwan, §5.1. I will take the point of view of Miranda [5], pp. 33-35, and Donaldson, Chapter 6. I will look more at θ functions than at the Weierstrass $\mathfrak{p}$-function.

8. Max Noether's Fundamental Theorem and Divisors

Thursday, April 12th.

We switch back to a more algebraic point of view. This is the first lecture on the Riemann-Roch Theorem. The material is covered in Fulton [3], Chapter 5, §5, p. 119-122, and then Chapter 8, §1, p. 187.

9. The Riemann-Roch Theorem

Tuesday, April 17th.

The Riemann-Roch Theorem is the most important theorem in the theory of curves. Kirwan only covers the case of smooth planes curves: §6.3. It is only slightly more difficult to cover the case of plane curves with only ordinary singular points, which effectively gives a proof for all curves. I will present the classical proof of Brill and Noether, largely as given in Fulton, pp. 192-212, replacing the material on differential forms with what we did earlier with differential forms.

10. Applications of the Riemann-Roch Theorem

Thursday, April 19th.

One important application is the existence of a group structure on a smooth cubic curve: Kirwan Theorem 6.39, p. 163. Many others are given in Miranda, Chapter VII and Fulton.

11. Resolution of Plane Singularities

Tuesday, April 24th.

An outline of the resolution process: Kirwan §7.1.

12. Newton polygons and Puiseux Expansions

Thursday, April 26th.

Kirwan §7.2. Last day of class.

References

[1] Simon Donaldson, *Riemann Surfaces*, Oxford Graduate Texts in Mathematics 22, Oxford, 2011.
[2] David S. Dummit, Richard M. Foote, *Abstract Algebra*, Second Edition, Prentice Hall, Upper Saddle River, 1999.
[3] William Fulton, *Algebraic Curves*, Benjamin, New York, 1969.
[4] Frances Kirwan, *Complex Algebraic Curves*, Cambridge U. P., Cambridge, 1992.
[5] Rick Miranda, *Algebraic Curves and Riemann Surfaces*, Graduate Studies in Mathematics 5, AMS, Providence, RI, 1995.
[6] James R. Munkres, *Topology*, Second Edition, Prentice Hall, 2000.
[7] B. L. van der Waerden, *Modern Algebra*, Frederick Ungar, New York, 1949.
[8] Robert J. Walker, *Algebraic Curves*, Dover, New York, 1962.