MATH V2020 PROBLEM SET 8
DUE NOVEMBER 18, 2008.

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Note: problem 1 corrected to work out more nicely.

(1) Consider the system of differential equations

$$\begin{aligned} y_1' &= -3y_1 + 2y_2 \\ y_2' &= -2y_1 + 2y_2 \end{aligned}$$

subject to the initial conditions $y_1(0) = 1$, $y_2(0) = 3$.

  (a) Solve the system by decoupling it (the first method we used in class).
  (b) Check that your solution is, indeed, a solution.
  (c) Solve the system using the matrix exponential.
  (d) Check that your two solutions agree.

(2) Write the differential equation $y''' = 5y'' - y' + 5y$ as a system of first-order differential equations.

(3) Exponentiating JNF matrices, I. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

  (a) Compute A^2, A^3, A^4.
  (b) Give a formula for A^n. Prove your answer by induction.
  (c) What is e^A?
  (d) What is e^{tA}? (*Be careful.*)
  (e) Consider the system of differential equations $y' = Ay$. Use matrix exponentiation to find the solution subject to the initial conditions $y(0) = (1, 3)^T$. Verify that what you found is, in fact, a solution.

(4) Exponentiating JNF matrices, II. Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

  (a) Compute A^2, A^3, A^4.
  (b) Give a formula for A^n. You don't have to prove your answer this time.
  (c) What is e^A?
  (d) What is e^{tA}? (*Be careful.*)

(5) Exponentiating JNF matrices, III. Let

$$A = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

be a Jordan block. What is e^A? e^{tA}? Justify your answers.

(6) Consider the system of differential equations

$$y_1'(t) = y_1(t) + y_2(t)$$
$$y_2'(t) = -y_1(t) + 3y_2(t).$$

Use matrix exponentiation to find the solution satisfying $y_1(0) = 1$, $y_2(0) = 2$. (Hint: put the corresponding matrix in JNF.)

(7) Lengths and angles.

(a) On $\mathbb{R}^3$ with its usual dot product, compute the lengths of $(1, 1, 1)^T$ and $(1, 2, 3)^T$, and the angle between them. (Not cooked to come out nicely.)

(b) On $\mathcal{C}^\infty[0, 1]$ with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$, compute the lengths of $f(x) = x$ and $g(x) = \sin(2\pi x)$, and the angle between them. (Also not cooked.)

(8) Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Use the Gram-Schmidt process to find an orthonormal basis for the image (column space) of A, with respect to the standard dot product on $\mathbb{R}^3$.

(9) Define an inner product $\langle \cdot, \cdot\rangle$ on $\mathcal{P}_{\leq 2}$ by $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)dx$.

(a) Prove that $\langle \cdot, \cdot\rangle$ does, in fact, give an inner product.

(b) Apply the Gram-Schmidt process to the basis $[2, x, x^2]$ to obtain an orthonormal basis for $\mathcal{P}_{\leq 2}$ (with respect to this inner product).

(10) Let $S = \{v_1, \dots, v_k\} \subset V$ be a set of vectors in an inner product space V. Prove: if the vectors in S are orthonormal then S is linearly independent.

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