MATH V2020 PROBLEM SET 7
DUE OCTOBER 28, 2008.

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(1) Some proofs by induction.
   (a) Prove by induction: $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$.
   (b) Prove by induction: If A is an upper triangular matrix then $\det(A)$ is the product of the diagonal entries of A. (You gave an explanation in a previous homework. Now turn that into a rigorous proof. Your proof should only be a few sentences long.)

(2) More diagonalization. Find a matrix P so that $P^{-1}\begin{pmatrix} -3 & 5 \\ -1 & 1 \end{pmatrix}P$ is diagonal.

(3) Let $F : \mathcal{P}_{\leq 1} \to \mathcal{P}_{\leq 1}$ be given by

$$F(p(x)) = 3p(0) + p(x) - xp(x) + 2p'(x) + x^2p'(x).$$

(The funny choice of F is, hopefully, carefully cooked.)
   (a) Let $\mathcal{B} = [1, x]$, a basis for $\mathcal{P}_{\leq 1}$. Find the matrix for F with respect to $\mathcal{B}$.
   (b) Find the eigenvalues λ_1 and λ_2 of F.
   (c) Find an eigenvector v_i corresponding to each eigenvalue λ_i. Write your eigenvectors as elements of $\mathcal{P}_{\leq 1}$, i.e., polynomials.
   (d) Check directly that your eigenvectors are, in fact, eigenvectors of F, by applying F to them.
   (e) Find a basis for $\mathcal{B}$ with respect to which F is represented by a diagonal matrix. What is the matrix for F with respect to this basis? (Hint: this should be easy from what you've already done.)

(4) Prove: if A is an $n \times n$ matrix with real entries, and $\lambda \in \mathbb{C}$ is an eigenvalue of A then $\overline{\lambda}$ is also an eigenvalue of A. (Here, $\overline{\lambda}$ is the complex conjugate of λ. i.e., $\overline{a + bi} = a - bi$.)

(5) Find the transition (Markov) matrix corresponding to the following weighted graph from Figure 1. For Evan's grading convenience, order the vertices with A first, B second, and so on when you're forming the matrix. Optional: use a computer to find a steady-state vector for this process (by computing A to a high power). Or: make up a story for a process leading to this graph. (I didn't have one in mind when I drew it. Extra points for creativity.) Or: find a way of changing one pair of weights from one vertex in the graph so that in the new graph there's an obvious steady-state vector. (Hint: it's okay to set a weight to zero.)

(6) Verify the Cayley-Hamilton theorem (that $P_A(A) = 0$) for the matrix $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.

(7) For each of the following matrices, find the eigenvalues. Find their algebraic and geometric multiplicities. Find the Jordan normal form of each matrix. (Note: I am *not* asking you to find a change of basis matrix P so that $P^{-1}AP$ is in Jordan normal form. I just want to know what the form itself is. e.g., if A is diagonal, you just need to find the eigenvalues, not the eigenvectors. But do justify your answer.)

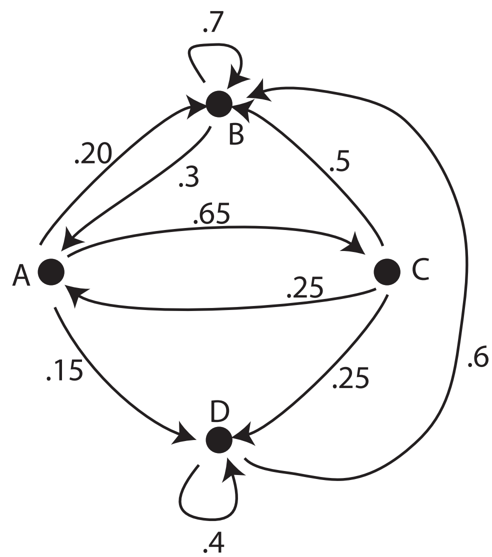

FIGURE 1. A weighted graph corresponding to a Markov process.

(a) $A = \begin{pmatrix} 3 & 4 \\ -1 & 0 \end{pmatrix}$
(b) $B = \begin{pmatrix} 6 & -2 \\ 6 & -1 \end{pmatrix}$
(c) $C = \begin{pmatrix} 8 & 9 & 0 \\ -4 & -4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(8) Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix $A = \begin{pmatrix} -3 & 4 & -4 \\ -18 & 20 & -22 \\ -9 & 11 & -13 \end{pmatrix}$ with respect to the standard basis.
- (a) The eigenvalues of A are 3 and -2. What are their algebraic multiplicities? Their geometric multiplicities? (Suggestion: compute the characteristic polynomial.)
- (b) Find a generalized eigenvector v_1 of eigenvalue 3 and longevity 2. Compute $v_2 = (A - 3I)(v_1)$. What is $F(v_2)$?
- (c) Find an eigenvector v_3 of eigenvalue -2.
- (d) What is the matrix for F with respect to the basis $[v_1, v_2, v_3]$?
- (e) Find a matrix P so that $P^{-1}AP$ is in Jordan Normal Form. (This part shouldn't require any work, given what you've already done.)

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