MATH V2020 PROBLEM SET 2
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(1) Fill in the blanks in the proofs on page 3.

(2) Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+2y \\ -6x-4y \end{pmatrix}$$

  (a) Find a basis for the kernel of F. (This boils down to solving a very simple system of linear equations.)
  (b) Find a basis for the image of F.
  (c) Draw the image of F.
  (d) Draw the set of solutions of $F(x,y)^T = (0,0)^T$.[1] On the same graph, draw the set of solutions of $F(x,y)^T = (3,-6)^T$. Notice anything?
  (e) How many solutions are there to $F(x,y)^T = (1,1)^T$? What does this have to do with the image of F?
  (f) Find the matrix for F with respect to the standard basis for $\mathbb{R}^2$.

(3) Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y+8z \\ y+2z \\ x+y+5z \end{pmatrix}$$

  (a) Find a basis for the kernel of F. (This boils down to solving a system of linear equations.) Draw the kernel of F in $\mathbb{R}^3$.
  (b) Find a basis for the image of F. Draw the image of F in $\mathbb{R}^3$. (You will not be graded for neatness.)
  (c) Plot the set of solutions to $F(x,y,z) = (2,0,1)^T$ in $\mathbb{R}^3$.
  (d) Find the matrix for F with respect to the standard basis for $\mathbb{R}^3$.

(4) Prove that the kernel of a linear transformation is a vector subspace. (Your proof should start "Let V and W be vector spaces, and $F\colon V \to W$ a linear transformation." The whole proof should be quite short.)

(5) Let V be a real vector space and U_1, U_2 linear subspaces of V. Then $U_1 \cup U_2$ is a linear subspace of V is and only if either $U_1 \subset U_2$ or $U_2 \subset U_1$. Either
  - Prove this statement or
  - Draw several pictures in $\mathbb{R}^2$ and/or $\mathbb{R}^3$ indicating why it's true, and give a short explanation in words of why it's true.

(6) Matrices for reflections in $\mathbb{R}^2$.
  (a) Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection across the line $y = x$. Find the matrix for F with respect to the standard basis for $\mathbb{R}^2$.

[1] The notation $(x,y)^T$ is shorthand for $\begin{pmatrix} x \\ y \end{pmatrix}$.

   (b) Find a basis for $\mathbb{R}^2$ with respect to which the matrix for F is diagonal. (Hint: there is a vector v so that $F(v) = v$. There's another vector w so that $F(w) = -w$.)
   (c) Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection across the line through the origin making an angle θ with the x axis. Find the matrix for F with respect to the standard basis for $\mathbb{R}^2$. (This takes some drawing.)
   (d) Find a basis for $\mathbb{R}^2$ with respect to which the matrix for F is diagonal.

(7) Matrices for some maps between other vector spaces:
   (a) Define a map $F\colon \mathcal{P}_{\leq 3} \to \mathcal{P}_{\leq 2}$ by $F(p(x)) = p'(x) + p(2) + 3xp''(x)$. Find the matrix for F with respect to the bases $[1, x, x^2, x^3]$ for $\mathcal{P}_{\leq 3}$ and $[1, x, x^2]$ for $\mathcal{P}_{\leq 2}$.
   (b) Recall that $\mathcal{C}^0(\mathbb{R})$ denotes the vector space of continuous functions $\mathbb{R} \to \mathbb{R}$. Let $V = \mathrm{Span}\{\sin(x), \cos(x)\} \subset \mathcal{C}^0(\mathbb{R})$. Let $F\colon V \to V$ be defined by $F(f(x)) = f'(x)$. Find the matrix for F with respect to the basis $\mathcal{B} = [\sin(x), \cos(x)]$.
   (c) With notation as in part 7b, let $G\colon V \to V$ be the linear transformation $G(f(x)) = f''(x)$. Find the matrix for G with respect to the basis $\mathcal{B}$.

(8) Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ denote rotation about the line $x = y = z$ by 90 degrees. Find a basis $\mathcal{B}$ for $\mathbb{R}^3$ for which it is easy to express F in terms of a matrix, and find the matrix for F with respect to your basis $\mathcal{B}$.

(9) This problem is **optional** (because it's confusing).

   The set of linear transformations from V to W is itself a vector space: given linear transformations F and G from V to W, define $F + G$ by $(F + G)(v) = F(v) + G(v)$, and for $\lambda \in \mathbb{F}$ define (λF) by $(\lambda F)(v) = \lambda F(v)$. Let $\mathrm{Hom}(V, W)$ denote the vector space of linear transformations from V to W.[2]
   (a) Let $v \in V$. Then there is a map $E_v : \mathrm{Hom}(V, W) \to W$ defined by $E_v(F) = F(v)$. Prove that, for each v, the map E_v is a linear transformation.
   (b) What is the dimension of $\mathrm{Hom}(V, W)$, in terms of the dimensions of V and W? (Hint: think about matrices.)

[2]Hom stands for *homomorphism*, which is another word for linear transformation.

We will use the following lemma, which is proved using the Steinitz exchange trick (in exactly the way we did in class):

Lemma 1. *Let V be an n-dimensional vector space. Then any set of linearly independent vectors in V can have at most n elements.*

The following is sometimes called the *basis extension theorem.*

Theorem 1. *Let V be a finite-dimensional vector space, and S a linearly independent subset of V. Then S is contained in a basis for V.*

Proof. Write $S = \{v_1, \dots, v_k\}$. If $\text{Span}(S) = V$ then S is a __________________ for V and we're done. Otherwise, there is some vector $v \in V$ such that $v \notin \text{Span}(S)$. Let $S_2 = S \cup \{v\}$. We claim that S_2 is linearly __________________. Suppose not. Then there are numbers $a_1, \dots, a_k, a \in \mathbb{F}$ such that ____________________________________ If $a \neq 0$ then we have $v =$ ____________________________________ So, v lies in the __________________ of $v_1, \dots, v_k$, which is a contradiction.

So, $a = 0$. But then

$$a_1 v_1 + \cdots + a_k v_k = 0.$$

Since S is __________________, all of the a_i must be zero.

Now, repeat this process with __________________ in place of S. Either $\text{Span}(S_2) = V$, in which case S_2 is a __________________ for V and we're done; or, we can find a still larger set S_3 which is still __________________. Repeat. By Lemma 1, any set of linearly __________________ vectors in V can have at most $\dim(V)$ elements, the process must eventually terminate. But the only way it terminates is if one of the S_n is a basis for V. □

Corollary 2. *Let $F \colon V \to W$ be a linear transformation, with V finite-dimensional. Then*

$$\dim(\ker(F)) + \dim(\text{Im}(F)) = \dim(V).$$

Proof. Let $\{e_1, \dots, e_k\}$ be a basis for $\ker(F)$. By the basis extension theorem, we can find vectors $f_1, \dots, f_l$ so that $\{e_1, \dots, e_k, f_1, \dots, f_l\}$ is a basis for V. We claim that $\{F(f_1), \dots, F(f_l)\}$ is a basis for $\mathrm{Im}(F)$. To see this, we must show that $\{F(f_1), \dots, F(f_l)\}$ ________________ the image of F and are linearly ________________.

First, we prove that $\{F(f_1), \dots, F(f_l)\}$ span $\mathrm{Im}(F)$. Indeed, if $w \in \mathrm{Im}(F)$ then $w = f(v)$ for some v in V. Since $\{e_1, \dots, e_k, f_1, \dots, f_l\}$ ________________ V, there are numbers $a_1, \dots, a_k, b_1, \dots, b_l \in \mathbb{F}$ such that

$$v = a_1 e_1 + \cdots + a_k e_k + b_1 f_1 + \cdots + b_l f_l.$$

But then

$$\begin{aligned} w = F(v) &= \underline{\hspace{6cm}} \\ &= \underline{\hspace{6cm}} \\ &= b_1 F(f_1) + \cdots + b_l F(f_l) \end{aligned}$$

since $F(e_i) = 0$. So, w is in the span of $\{F(f_1), \dots, F(f_l)\}$.

Next, we prove that $F(f_1), \dots, F(f_l)$ are linearly ________________. Suppose that

$$a_1 F(f_1) + \cdots + a_l F(f_l) = 0.$$

Then,

$$\underline{\hspace{6cm}}.$$

So, $a_1 f_1 + \cdots + a_l f_l$ is in the ________________ of F. So, $a_1 f_1 + \cdots + a_l f_l = b_1 e_1 + \cdots + b_k e_k$ for some $b_1, \dots, b_k \in \mathbb{F}$, since $e_1, \dots, e_k$ ________________ $\ker(F)$. But then $(-b_1)e_1 + \cdots + (-b_k)e_k + a_1 f_1 + \cdots + a_l f_l = 0$. So, since $\{e_1, \dots, f_l\}$ is ________________, all a_i and b_j are zero. So, $F(f_1), \dots, F(F_l)$ are linearly ________________.

Since ________________ is a basis for $\mathrm{Im}(F)$, it follows that the dimension of $\mathrm{Im}(F)$ is l. Since ________________ is a basis for $\ker(F)$, the dimension of the kernel of F is k. And, since $\{e_1, \dots, e_k, f_1, \dots, f_l\}$ is a basis for ________________, the dimension of ________________ is $k + l$. This proves the corollary. □

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