MATH V2020 PROBLEM SET 10
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(1) Let $(V, \langle \cdot, \cdot \rangle$ be a finite-dimensional inner product space and $S \subset V$ a subspace. Prove that $(S^\perp)^\perp = S$.

Hint. Prove that $S \subset (S^\perp)^\perp$. Then use the fact that $\dim(V) = \dim(S) + \dim(S^\perp)$ and $\dim(V) = \dim(S^\perp) + \dim((S^\perp)^\perp)$.

(2) Let

$$A = \begin{pmatrix} 25 & -4 & -4 \\ -4 & 10 & 1 \\ -4 & 1 & 10 \end{pmatrix}.$$

Find an orthogonal matrix P such that $P^T AP$ is diagonal. What is $P^T AP$?

(Note: some of the numbers get pretty big, but the problem is cooked.)

(3) An $n \times n$ matrix A is called *skew-Hermitian* if $A^H = -A$.

If $(V, \langle \cdot, \cdot \rangle)$ is a Hermitian inner product space then a linear transformation $F \colon V \to V$ is called *skew-adjoint* if for any $v, w \in V$, $\langle F(v), w \rangle = -\langle v, F(w) \rangle$.

(a) Give an example of a skew-Hermitian matrix.
(b) Prove that if $F \colon V \to V$ is skew-adjoint and $\mathcal{B} = [e_1, \dots, e_n]$ is an orthonormal basis for V then the matrix for F with respect to $\mathcal{B}$ is skew-Hermitian.
(c) Prove that if F is skew-adjoint then the eigenvalues of F are all purely imaginary (i.e., of the form ir for some $r \in \mathbb{R}$).
(d) Prove that if F is skew-adjoint and v, w are eigenvectors of F of eigenvalues λ and μ respectively, with $\lambda \neq \mu$ then v and w are orthogonal.
(e) Prove that if F is skew-adjoint, v is an eigenvector of F and w is any vector so that $w \perp v$ then $F(w) \perp v$.
(f) Prove by induction that if F is skew-adjoint then there is an orthonormal basis $\mathcal{B}$ for V so that $[F]_\mathcal{B}$ is diagonal.

Hint for all parts. All of the proofs in this problem are very similar to proofs we did in class, for analogous properties of self-adjoint transformations.

(4) An $n \times n$ matrix with real entries is called *skew-symmetric* if $A^T = -A$. In this exercise, you will prove that if A is skew-symmetric then e^A is orthogonal.

(a) Find matrices A and B so that $e^{A+B} \neq e^A e^B$.
(b) Explain why if $AB = BA$ then $e^{A+B} = e^A e^B$. (Hint: each side gives you an infinite sum. Figure out what the coefficient of $A^m B^n$ is in each sum.)
(c) Explain why if A is skew-symmetric then $AA^T = A^T A$ (very short).
(d) Prove that if A is skew-symmetric then e^A is orthogonal. (Hint: this is not hard from parts (4b) and (4c). Compute $(e^A)(e^{A^T})$.)

Remark. In very fancy terminology, this statement is the fact that the *Lie algebra* of the orthogonal group is the space of skew-symmetric matrices.

Remark. It's also true that if A is skew-Hermitian then e^A is unitary. The proof is essentially the same.