MATH V2020 PROBLEM SET 1
DUE SEPTEMBER 9, 2008.

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Please keep track of how many hours of undistracted work this takes you. I'll ask you this on Tuesday, to help calibrate future assignments. (My target: roughly six hours per week.)

(This is really a foreign language assignment. And while there are a lot of problems, most of the solutions are quite short.)

[Problem 9 corrected in this version.]

(1) Let $S = \{x \in \mathbb{N} \mid x \text{ is prime}\}$, $T = \{x \in \mathbb{N} \mid x \text{ is odd}, x < 10\}$. List the elements of
   (a) $S \cap T$
   (b) $T \setminus S$
   (c) $(S \cap T) \times (T \setminus S)$.

(2) Let $p(x)$ be an even-degree polynomial with real coefficients. View $p(x)$ as a map $\mathbb{R} \to \mathbb{R}$. Explain why the map $p(x)$ is not bijective.

   (Note on interpreting the problem: the phrasing means you don't get to choose $p(x)$. You're supposed to prove that no matter what even-degree polynomial I give you, the corresponding map is not bijective.)

(3) Let $p(x) = x^3 + ax$. For which $a \in \mathbb{R}$ is $p(x)$
   (a) injective?
   (b) surjective?
   (c) bijective?

   (Justify, but don't necessarily prove, your answer.)

(4) Fill in the blanks in the proof on page 4.

(5) Using the axioms of vector spaces, give a two-column proof of the following:

   Lemma 1. *Let V be a real vector space, and x, y, z, w elements of V. Then*

$$((5(x + y)) + z) + w = (5x) + ((5y) + (z + w)).$$

   See Page 3 for an example of what I mean.

(6) Let $\mathcal{P}$ denote the set of all polynomials in the variable x, with real coefficients. Let $+$ be the usual operation of addition and define scalar multiplication by

$$\lambda(a_0 + \cdots + a_n x^n) = (\lambda a_0) + \cdots + (\lambda a_n) x^n.$$

   Recall that the degree of a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ with $a_n \neq 0$ is n. (For instance, the degree of $0x^3 + 3x^2 + x$ is 2.)
   (a) $\mathcal{P}$ is a vector space. Why? Verify a couple of the axioms.

(b) Let $\mathcal{P}_{\leq 5}$ be the subset of $\mathcal{P}$ of polynomials of degree *at most* 5. Is $\mathcal{P}_{\leq 5}$ a vector subspace of $\mathcal{P}$? Why or why not?

(c) Let $\mathcal{P}_{=5}$ be the subset of $\mathcal{P}$ of polynomials of degree *exactly* 5. Is $\mathcal{P}_{=5}$ a vector subspace of $\mathcal{P}$? Why or why not?

(7) Prove that $S = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$ is a vector subspace of $\mathbb{R}^2$ if and only if $b = 0$.

(Note: there are two parts. One starts "Suppose that $b = 0$.". The other starts either "Suppose that S is a vector subspace" or "Suppose that $b \neq 0$".)

(8) Let $\mathcal{P}$ denote the vector space of polynomials in x.

(a) *Prove* that $\mathcal{P}$ is not finite-dimensional, by finding an infinite number of linearly independent elements of $\mathcal{P}$. (This might be tricky because it's so easy.)

(b) Find a finite-dimensional subspace of $\mathcal{P}$ and give its dimension. (Yes, lots of easy choices here.)

(9) (a) Find a basis for $V = \{(x, y) \in \mathbb{R}^2 \mid 2x + 4y = 0\}$. What is the dimension of V?

(b) Find a basis for $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 4z = 0\}$. (This takes a little work.) What is the dimension of W?

(c) The vector $(10, -1, -2)$ is an element of W. Write $(10, -1, -2)$ as a linear combination of your basis vectors.

(10) For $a \in \mathbb{R}$, let $V = \mathrm{Span}\left\{\begin{pmatrix} a \\ a \\ b \end{pmatrix}\right\}$. Let $W = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right\}$.

(a) For $a = b = 1$, what are $\dim(V)$, $\dim(W)$, $\dim(V \cap W)$ and $\dim(V + W)$? Check that your answer is consistent with Theorem 3, p. 49.

(b) For $a = 1$ and $b = 0$, what are $\dim(V)$, $\dim(W)$, $\dim(V \cap W)$ and $\dim(V + W)$? Check that your answer is consistent with Theorem 3, p. 49.

(c) For $a = b = 0$, what are $\dim(V)$, $\dim(W)$, $\dim(V \cap W)$ and $\dim(V + W)$? Check that your answer is consistent with Theorem 3, p. 49.

Sample two-column proof:

Lemma 2. *Let V be a real vector space and x, y elements of V. Then $2(x+y) + 3(x+y) = (5x) + (5y)$.*

Proof.

(1)	$2(x+y) = 2x + 2y$	Distributivity
(2)	$3(x+y) = 3x + 3y$	Distributivity
(3)	$2(x+y) + 3(x+y) = (2x+2y) + (3x+3y)$	Steps (1) and (2)
(4)	$(2x+2y) + (3x+3y) = (2x + (2y+3x)) + 3y$	Associativity for vector addition
(5)	$(2x + (2y+3x)) + 3y = (2x + (3x+2y)) + 3y$	Commutativity for vector addition
(6)	$(2x + (3x+2y)) + 3y = (2x+3x) + (2y+3y)$	Associativity for vector addition
(7)	$(2x+3x) + (2y+3y) = (2+3)x + (2+3)y$	Distributivity (twice)
(8)	$(2+3)x + (2+3)y = 5x + 5y$	(Arithmetic in $\mathbb{R}$; see below.)
(9)	$2(x+y) + 3(x+y) = 5x + 5y$	Steps (3), (4), (5), (6), (7) and (8) and the transitive property of equality (5 times)

□

(Yes, I know this is a pain: I had to typeset this whole monster. I wont ask this of you after this problem set. The point is that, in principle, any proof can be reduced to a sequence of steps like this. For what it's worth, I remember being surprised that the short list of axioms we have really were enough to do any manipulation like this.)

Remark. We haven't discussed axioms for arithmetic in $\mathbb{Z}$ or $\mathbb{R}$, so let's not make a big deal of it. In the usual way of axiomatizing arithmetic of $\mathbb{Z}$, the proof that $2+3 = 5$ would boil down to $(1+1)+((1+1)+1) = (((1+1)+1)+1)+1$, by a sequence of applications of the associative law. (Two is defined to be $1+1$, and 3 to be $((1+1)+1)$.) But that's for another course.

Theorem 1. *Let X, Y and Z be sets. If $f\colon X \to Y$ is injective and $g\colon Y \to Z$ is injective then $g \circ f\colon X \to Z$ is injective.*

Proof. Suppose that $x_1, x_2 \in X$ are such that $g \circ f(x_1) = g \circ f(x_2)$. We will show that ____________________. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $g(y_1) =$ ____________________. So, since g is injective, ____________________. So, $f(x_1) =$ ____________________. So, since ____________________, $x_1 = x_2$, as desired. □

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