MATH W4052 PROBLEM SET 5
DUE FEBRUARY 23, 2011.

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(1) Let $X_\bullet$ and $Y_\bullet$ be simplicial complexes. Define the disjoint union $X_\bullet \amalg Y_\bullet$ of $X_\bullet$ and $Y_\bullet$. (Hint: this is easy.) What is $H_n(X_\bullet \amalg Y_\bullet)$? Briefly explain why.

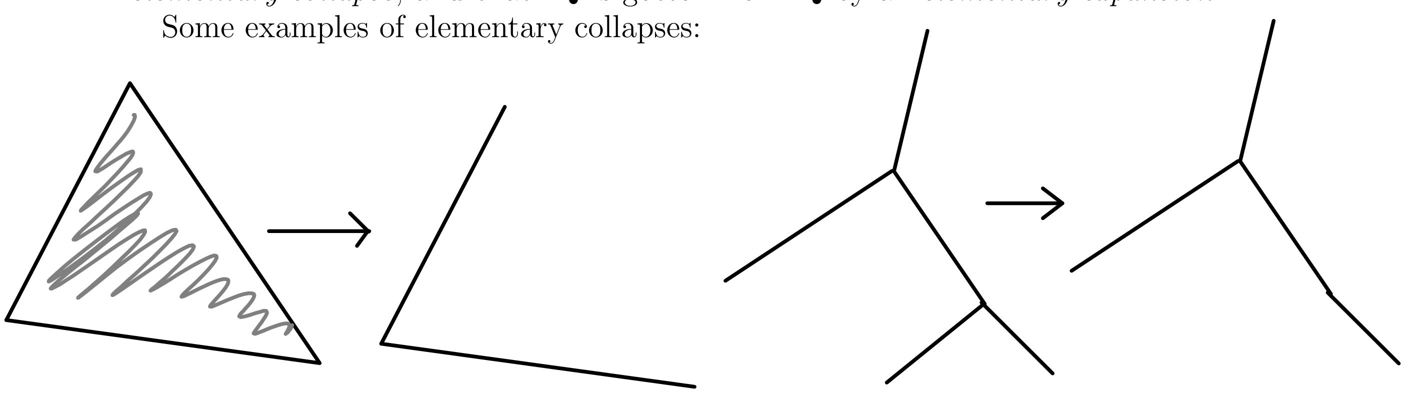
(2) Let $X_\bullet$ and $Y_\bullet$ be simplicial complexes. Define the wedge sum $X_\bullet \vee Y_\bullet$ of $X_\bullet$ and $Y_\bullet$, the result of gluing $X_\bullet$ to $Y_\bullet$ at a single vertex. (The result depends somewhat on the choice of vertex.) What is $H_n(X_\bullet \vee Y_\bullet)$? Briefly explain why.

(3) Prove that the geometric realization of a simplicial map is well-defined, i.e., respects the equivalence relation $\sim$.

(4) In this exercise you will prove invariance of simplicial homology under *simple homotopy equivalence*. (Most (probably all) of the homotopy equivalences you've seen are simple homotopy equivalences, so this is pretty good.)

We start with a definition. Let $X_\bullet$ be a simplicial complex. An n-simplex $\{v_0, \dots, v_n\} \in X_n$ is *free* if it is the face of exactly one $(n+1)$ simplex $\{v_0, \dots, v_n, v_{n+1} \in X_{n+1}\}$ (i.e., there is a unique vertex v_{n+1} so that $\{v_0, \dots, v_n, v_{n+1}\} \in X_{n+1}$). Let $Y_\bullet$ be the simplicial complex given by $Y_i = X_i$ if $i \neq n, n+1$, $Y_n = X_n \setminus \{\{v_0, \dots, v_n\}\}$ and $Y_{n+1} = X_{n+1} \setminus \{\{v_0, \dots, v_{n+1}\}\}$. Then we say that $Y_\bullet$ is gotten from $X_\bullet$ by an *elementary collapse*, and that $X_\bullet$ is gotten from $Y_\bullet$ by an *elementary expansion*.

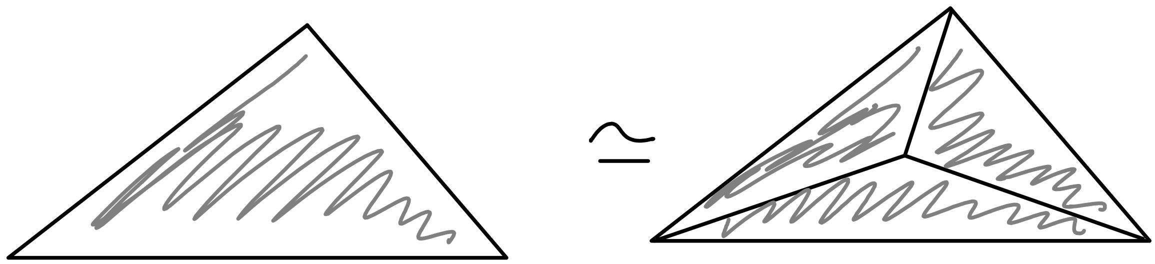
Some examples of elementary collapses:



If you can get from $X_\bullet$ to $Y_\bullet$ by a sequence of elementary collapses and elementary expansions then we say that $X_\bullet$ and $Y_\bullet$ are *simple homotopy equivalent*. For example, any tree is simple homotopy equivalent to a single point.

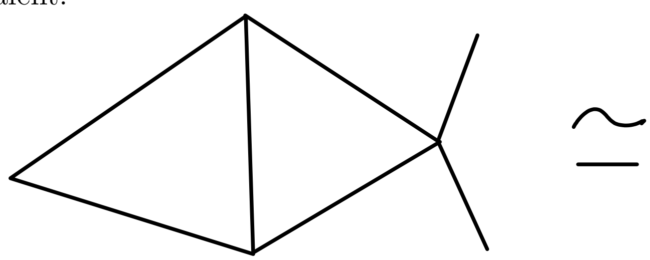
(a) Let S^1_n denote the simplicial complex for a circle with n vertices. Show that S^1_n is simple homotopy equivalent to S^1_{n+1}. (A sequence of pictures is enough.)

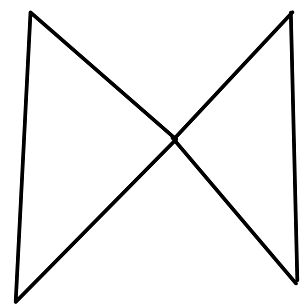
(b) Show that the following pair of simplicial complexes are simple homotopy equivalent:



(Again, a sequence of pictures suffices.)

(c) Show that the following pair of simplicial complexes are simple homotopy equivalent:





(Again, a sequence of pictures suffices.)

(d) Suppose that $Y_\bullet$ is obtained from $X_\bullet$ by an elementary collapse. Then there is an obvious simplicial map $f\colon Y_\bullet \to X_\bullet$. Prove that $f_*\colon H_n(Y_\bullet) \to H_n(X_\bullet)$ is an isomorphism for each n. Conclude that simplicial homology is invariant under simple homotopy equivalence.

(5) Prove the "simplicial Brouwer fixed point theorem": if $X_\bullet$ is a simplicial complex so that $|X_\bullet|$ is homeomorphic to $\mathbb{D}^n$ and $f\colon X_\bullet \to X_\bullet$ is a simplicial map then $|f|$ has a fixed point. You may assume that simplicial homology is independent of the triangulation and that $H_n(S^n) \cong \mathbb{Z}$ (for any n).

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