Towards bordered Heegaard Floer homology

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- We'll focus on $\widehat{\mathrm{HF}} = H_*(\widehat{\mathrm{CF}})$, the mapping cone of $U : \mathrm{CF}^+ \to \mathrm{CF}^+$.
- Conjecturally, $\mathrm{HF}^+ = \widecheck{HM} = ECH_*$.

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such that

- If $Y = Y_1 \cup_F Y_2$ then

$$\widehat{\mathrm{CF}}(Y) = \widehat{\mathrm{CFA}}(Y_1) \otimes_{\mathcal{A}(F)} \widehat{\mathrm{CFD}}(Y_2).$$

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Bordered Y^3, $\partial Y^3 = F$	compact, oriented 3-manifold with connected boundary, orientation-preserving homeomorphism $F \to \partial Y$	Right A_∞-module $\widehat{\mathrm{CFA}}(Y)$ over $\mathcal{A}(F)$, Left dg-module $\widehat{\mathrm{CFD}}(Y)$ over $\mathcal{A}(-F)$, well-defined up to homotopy.

Satisfying the pairing theorem:

Theorem

If $\partial Y_1 = F = -\partial Y_2$ *then*

$$\widehat{\mathrm{CF}}(Y_1 \cup_\partial Y_2) \simeq \widehat{\mathrm{CFA}}(Y_1) \widetilde{\otimes}_{\mathcal{A}(F)} \widehat{\mathrm{CFD}}(Y_2).$$

Further structure (in progress):

- To an $\phi \in \mathrm{MCG}(F)$, bimodules $\widehat{\mathrm{CFDA}}(\phi)$, $\widehat{\mathrm{CFAD}}(\phi)$.

$$\widehat{\mathrm{CFA}}(\phi(Y)) \simeq \widehat{\mathrm{CFA}}(Y)\widetilde{\otimes}_{\mathcal{A}(F)}\widehat{\mathrm{CFDA}}(\phi)$$
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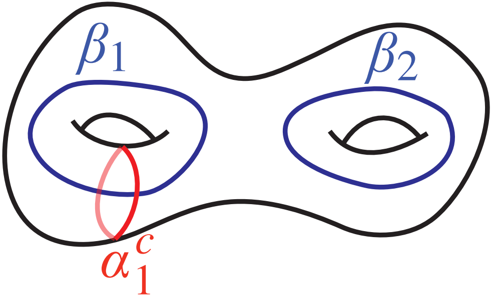
Bordered Heegaard diagrams

- Let $(\overline{\Sigma}_g, \alpha_1^c, \dots, \alpha_{g-k}^c, \beta_1, \dots, \beta_g)$ be a Heegaard diagram for a Y^3 with bdy.



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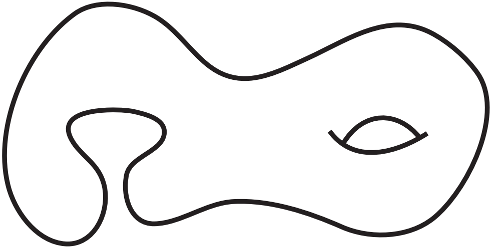

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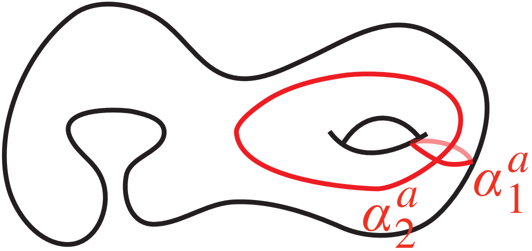

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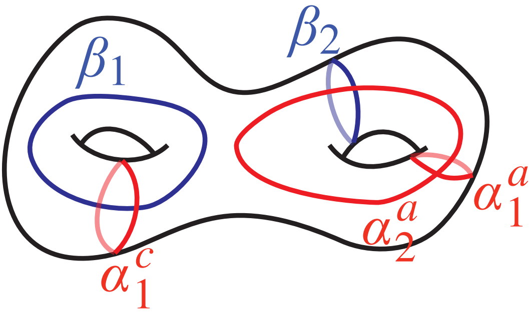
- Let $\alpha_1^a, \ldots, \alpha_{2k}^a$ be circles in $\Sigma' \setminus (\text{new disks}$ intersecting in one point p, giving a basis for $\pi_1(\Sigma')$.
- These give circles $\alpha_1^a, \ldots, \alpha_{2k}^a$ in $\overline{\Sigma}$.



- Let $\Sigma = \overline{\Sigma} \setminus \mathbb{D}_\epsilon(p)$.
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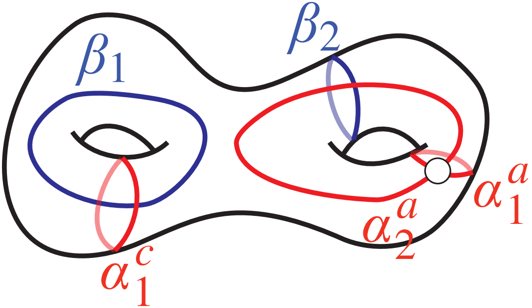

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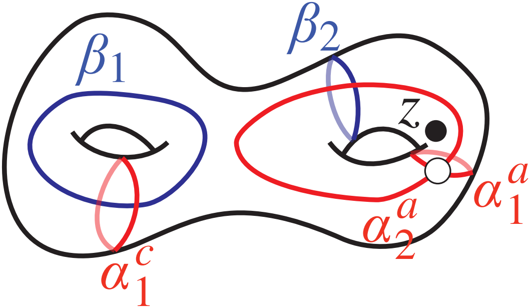
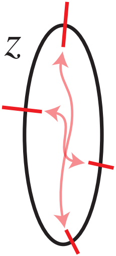
- Fix also $z \in \overline{\Sigma}$ near p.



A small circle near p looks like:



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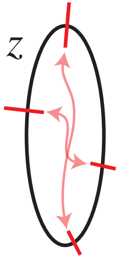
This is called a *pointed matched circle* $\mathcal{Z}$.



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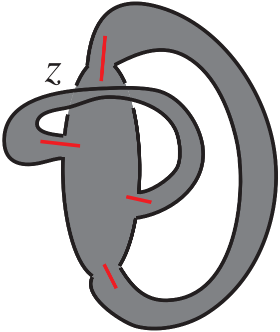

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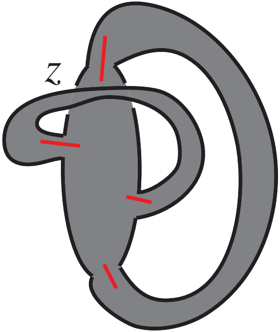
We will associate a *dg* algebra $\mathcal{A}(\mathcal{Z})$ to $\mathcal{Z}$.



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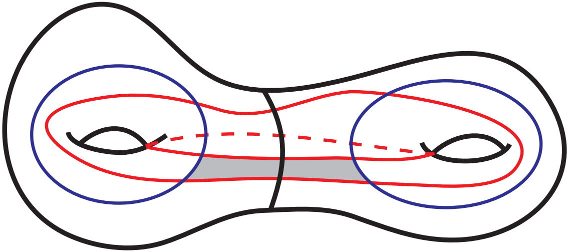
- Decomposing ordinary $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ into bordered H.D.'s $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1) \cup (\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$, would want to consider holomorphic curves crossing $\partial\Sigma_1 = \partial\Sigma_2$.



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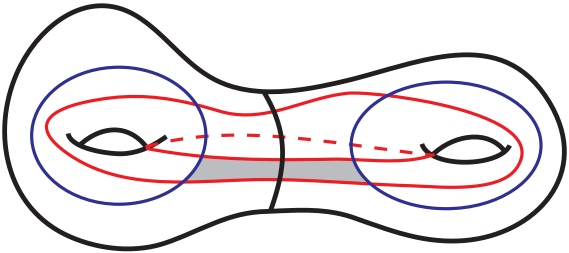

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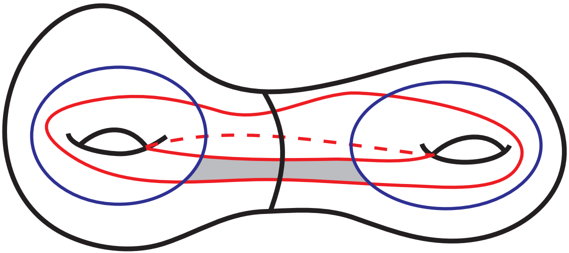
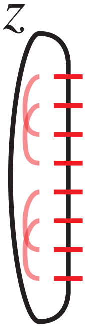
- Analyzing some simple models, in terms of *planar grid diagrams*, suggested the product and relations in the algebra.



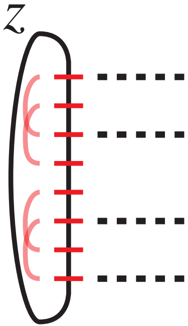
So...

- Let $\mathcal{Z}$ be a pointed matched circle, for a genus k surface.



So...

- Let $\mathcal{Z}$ be a pointed matched circle, for a genus k surface.
- Primitive idempotents of $\mathcal{A}(\mathcal{Z})$ correspond to k-element subsets I of the $2k$ pairs in $\mathcal{Z}$.
- We draw them like this:



- A pair (I, ρ), where ρ is a Reeb chord in $\mathcal{Z} \setminus z$ starting at I specifies an algebra element $a(I, \rho)$.
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More generally, given $(I, \boldsymbol{\rho})$ where $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_\ell\}$ is a set of Reeb chords starting at I, with:

- $i \neq j$ implies ρ_i and ρ_j start and end on different pairs.
- $\{\text{starting points of } \rho_i\text{'s}\} \subset I$.

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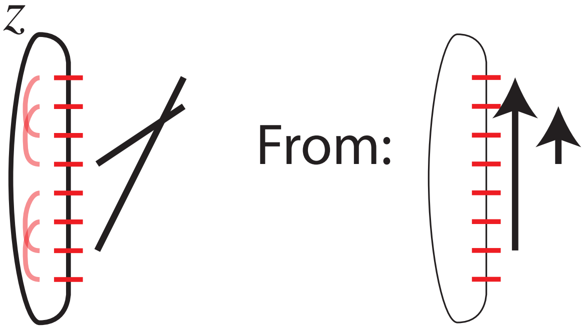
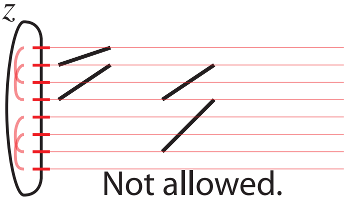


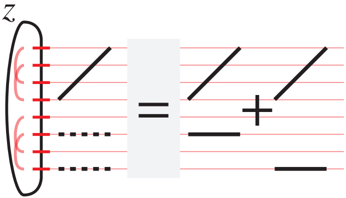
These generate $\mathcal{A}(\mathcal{Z})$ over $\mathbb{F}_2$.

That is, $\mathcal{A}(\mathcal{Z})$ is the subalgebra of the algebra of *k-strand, upward-veering flattened braids on $4k$ positions* where:

- no two start or end on the same pair

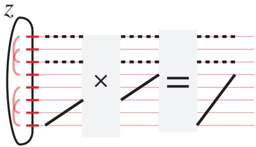


- Algebra elements are fixed by "horizontal line swapping".



Multiplication...

...is concatenation if sensible, and zero otherwise.



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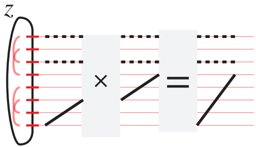

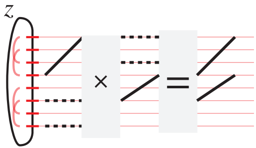

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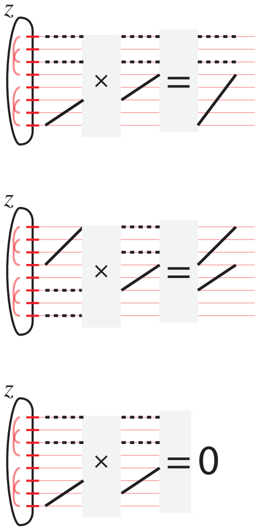
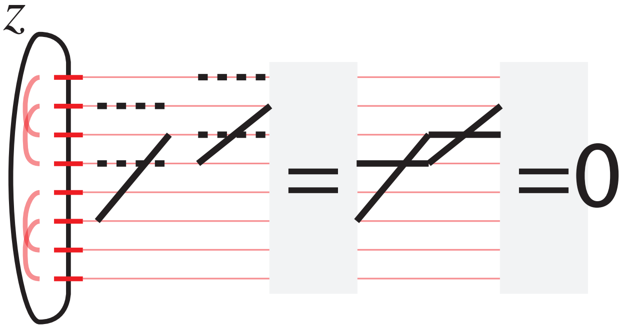


Double crossings

We impose the relation

$$(\text{double crossing}) = 0.$$

e.g.,

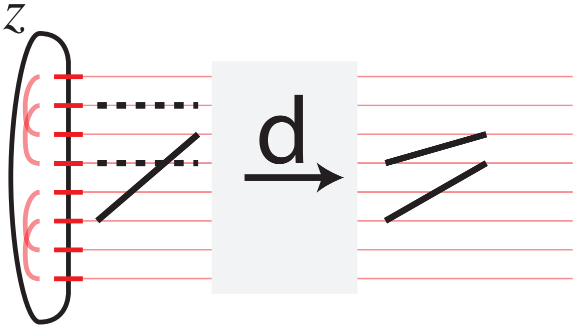


The differential

There is a differential d by

$$d(a) = \sum \text{smooth one crossing of } a.$$

e.g.,



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  - Multiplying consecutive Reeb chords concatenates them.
  - Far apart Reeb chords commute.
- The algebra is finite-dimensional over $\mathbb{F}_2$, and has a nice description in terms of flattened braids.

Gradings

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It was a good suggestion.

HM(Y) is graded by homotopy classes of nonvanishing vector fields on Y. So $\mathcal{A}(F)$ should be graded by homotopy classes of nonvanishing vector fields v on $F \times [0,1]$ such that

$$v|_{F\times\partial[0,1]} = v_0$$

for some given v_0.
(Think of $F \times [0,1]$ as a collar of ∂Y.)

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(Think of $F \times [0,1]$ as a collar of ∂Y.)
This is a group G under concatenation in $[0,1]$.

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- It follows that G is a $\mathbb{Z}$-central extension of $H_1(F)$,

$$0 \to \mathbb{Z} \to G \to H_1(F) \to 0.$$

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- Note: in the end, we define these gradings combinatorially, not geometrically.

The cylindrical setting for classical $\widehat{CF}$:

Fix an ordinary H.D. $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. (Here, $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$.)

- The chain complex $\widehat{CF}$ is generated over $\mathbb{F}_2$ by g-tuples $\{x_i \in \alpha_{\sigma(i)} \cap \beta_i\} \subset \boldsymbol{\alpha} \cap \boldsymbol{\beta}$. ($\sigma \in S_g$ is a permutation.) (cf. $T_\alpha \cap T_\beta \subset Sym^g(\Sigma)$.)

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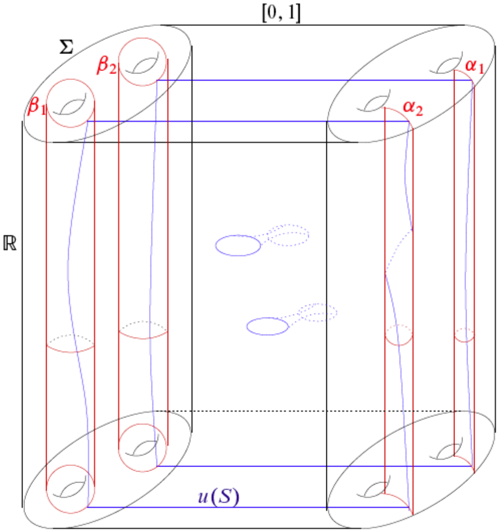
- The differential counts embedded holomorphic maps

$$(S, \partial S) \to (\Sigma \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times 1 \times \mathbb{R}) \cup (\boldsymbol{\beta} \times 0 \times \mathbb{R}))$$

  asymptotic to $\mathbf{x} \times [0,1]$ at $-\infty$ and $\mathbf{y} \times [0,1]$ at $+\infty$.
- For $\widehat{CF}$, curves may not intersect $\{z\} \times [0,1] \times \mathbb{R}$.

A useless schematic of a curve in $\Sigma \times [0,1] \times \mathbb{R}$.



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- The $e\infty$ asymptotics are *Reeb chords* $\rho_i \times (1, t_i)$.
- The asymptotics $\rho_{i_1}, \dots, \rho_{i_\ell}$ of u inherit a partial order, by $\mathbb{R}$-coordinate.

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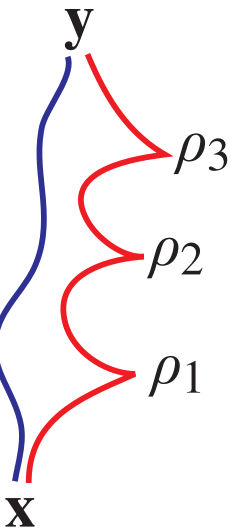
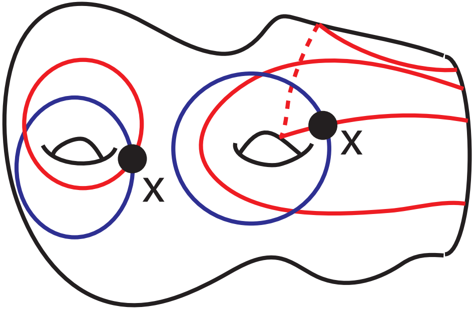


Generators of $\widehat{\text{CFD}}$...

Fix a bordered Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$
$\widehat{\text{CFD}}(\Sigma)$ is generated by g-tuples $\mathbf{x} = \{x_i\}$ with:

- one x_i on each β-circle
- one x_i on each α-circle
- no two x_i on the same α-arc.



Generators of $\widehat{\mathrm{CFD}}$...

Fix a bordered Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$

$\widehat{\mathrm{CFD}}(\Sigma)$ is generated by g-tuples $\mathbf{x} = \{x_i\}$ with:

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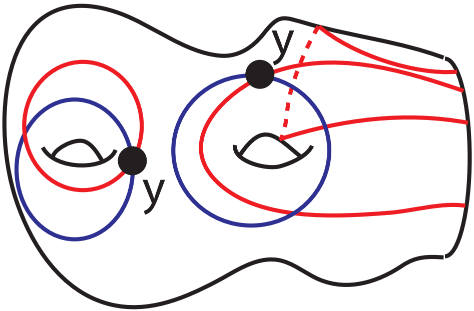
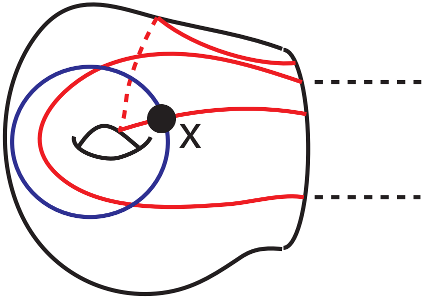


...and associated idempotents.

- To **x**, associate the idempotent $I(\mathbf{x})$, the α-arcs **not** occupied by **x**.



- As a left $\mathcal{A}$-module,

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...and associated idempotents.

- To $\mathbf{x}$, associate the idempotent $I(\mathbf{x})$, the α-arcs **not** occupied by $\mathbf{x}$.
- As a left $\mathcal{A}$-module,
$$\widehat{\mathrm{CFD}} = \oplus_{\mathbf{x}} \mathcal{A} I(\mathbf{x}).$$
- So, if I is a primitive idempotent, $I\mathbf{x} = 0$ if $I \neq I(\mathbf{x})$ and $I(\mathbf{x})\mathbf{x} = \mathbf{x}$.

The differential on $\widehat{CFD}$.

$$d(\mathbf{x}) = \sum_{\mathbf{y}} \sum_{(\rho_1,\dots,\rho_n)} (\#\mathcal{M}(\mathbf{x},\mathbf{y};\rho_1,\dots,\rho_n))\, a(\rho_1, I(\mathbf{x}))\cdots a(\rho_n, I_n)\mathbf{y}.$$

where $\mathcal{M}(\mathbf{x},\mathbf{y};\rho_1,\dots,\rho_n)$ consists of holomorphic curves asymptotic to

- $\mathbf{x}$ at $-\infty$
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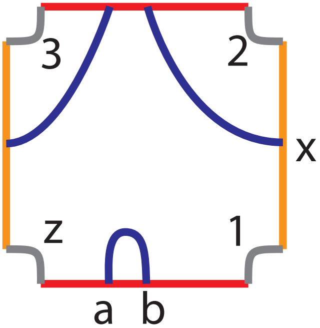
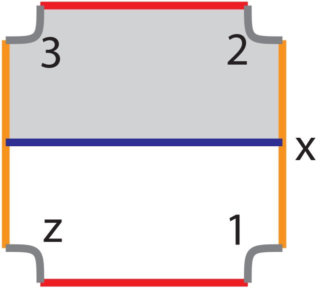
Example D1: a solid torus.



$$d(a) = b + \rho_3 x$$
$$d(x) = \rho_2 b$$
$$d(b) = 0.$$

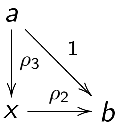
Example D2: same torus, different diagram.



$$d(\mathbf{x}) = \rho_2\rho_3\mathbf{x} = \rho_{23}\mathbf{x}.$$

Comparison of the two examples.

First chain complex:



Second chain complex:

$$x \xrightarrow{\rho_{23}} x$$

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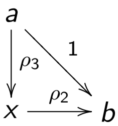

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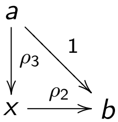

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Theorem

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $(\Sigma, \boldsymbol{\alpha}', \beta', z')$ are pointed bordered Heegaard diagrams for the same bordered Y^3 then $\widehat{\mathrm{CFD}}(\Sigma)$ is homotopy equivalent to $\widehat{\mathrm{CFD}}(\Sigma')$.

Generators and idempotents of $\widehat{\mathrm{CFA}}$.

Fix a bordered Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$
$\widehat{\mathrm{CFA}}(\Sigma)$ is generated by the same set as $\widehat{\mathrm{CFD}}$: g-tuples $\mathbf{x} = \{x_i\}$ with:

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...counts only holomorphic curves contained in a compact subset of Σ, i.e., with no asymptotics at $e\infty$.

The module structure on $\widehat{\mathrm{CFA}}$

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- Given a set $\boldsymbol{\rho}$ of Reeb chords, define

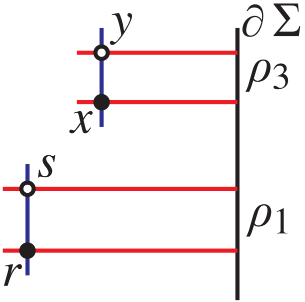
$$\mathbf{x} \cdot a(J(\mathbf{x}), \boldsymbol{\rho}) = \sum_{\mathbf{y}} (\#\mathcal{M}(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}))\, \mathbf{y}$$

where $\mathcal{M}(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho})$ consists of holomorphic curves asymptotic to

- $\mathbf{x}$ at $-\infty$.
- $\mathbf{y}$ at $+\infty$.
- $\boldsymbol{\rho}$ at $e\infty$, *all at the same height.*

A local example of the module structure on $\widehat{\mathrm{CFA}}$.

- Consider the following piece of a Heegaard diagram, with generators $\{r, x\}, \{s, x\}, \{r, y\}, \{s, y\}$.



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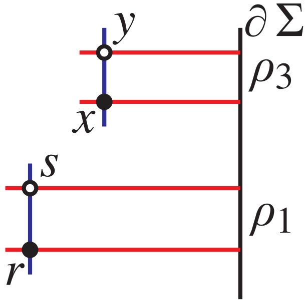

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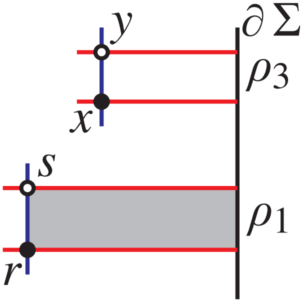

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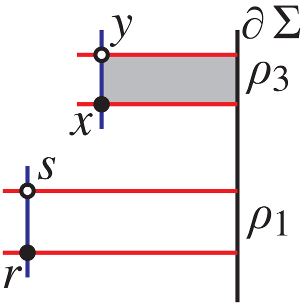

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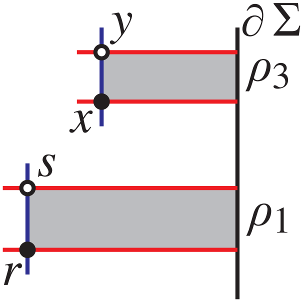
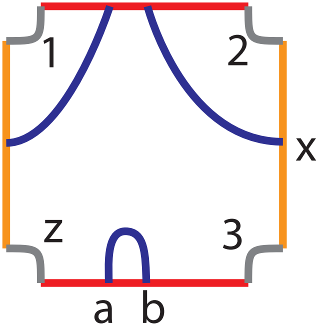


Example A1: a solid torus.



$$d(a) = b$$
$$a\rho_1 = x$$
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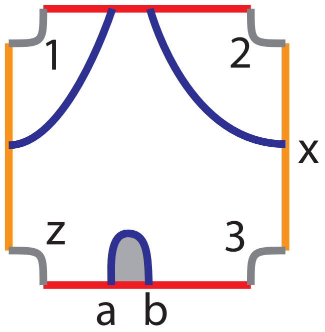

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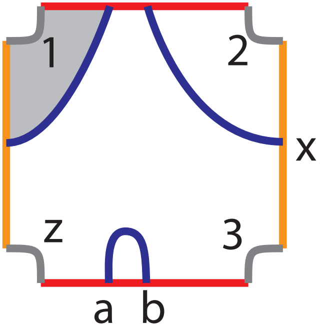

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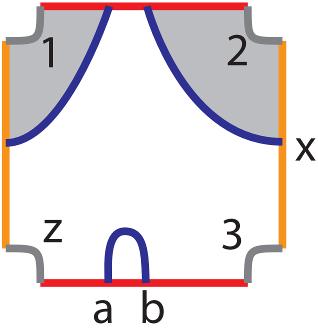

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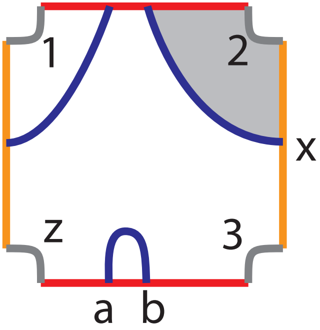

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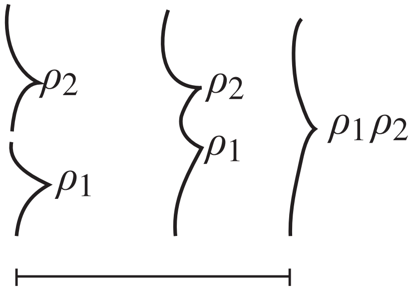
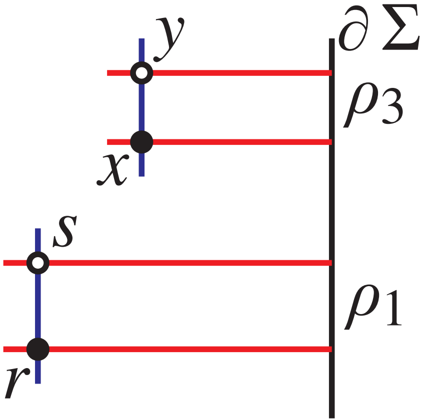
Why associativity should hold...

- $(\mathbf{x} \cdot \rho_i) \cdot \rho_j$ counts curves with ρ_i and ρ_j infinitely far apart.
- $\mathbf{x} \cdot (\rho_i \cdot \rho_j)$ counts curves with ρ_i and ρ_j at the same height.
- These are ends of a 1-dimensional moduli space, with height between ρ_i and ρ_j varying.

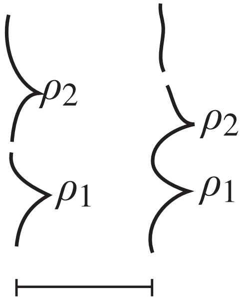


The local model again.



...and why it doesn't.

- But this moduli space might have other ends: broken flows with ρ_1 and ρ_2 at a fixed nonzero height.



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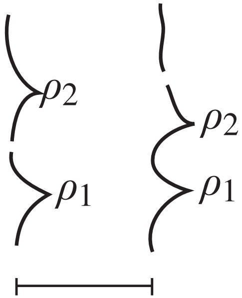
- But this moduli space might have other ends: broken flows with ρ_1 and ρ_2 at a fixed nonzero height.
- These moduli spaces – $\mathcal{M}(\mathbf{x}, \mathbf{y}; (\rho_1, \rho_2))$ – measure failure of associativity. So...



Higher A_∞-operations

Define

$$m_{n+1}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_n)) = \sum_{\mathbf{y}} (\#\mathcal{M}(\mathbf{x}, \mathbf{y}; (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n)))\, \mathbf{y}$$

where $\mathcal{M}(\mathbf{x}, \mathbf{y}; (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n))$ consists of holomorphic curves asymptotic to

- $\mathbf{x}$ at $-\infty$.
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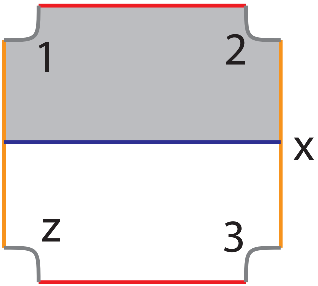
- $\boldsymbol{\rho}_1$ all at one height at $e\infty$, $\boldsymbol{\rho}_2$ at some other (higher) height at $e\infty$, and so on.

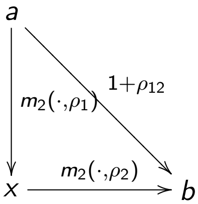
Example A2: same torus, different diagram.



$$m_3(x, \rho_2, \rho_1) = x$$
$$m_4(x, \rho_2, \rho_{12}, \rho_1) = x$$
$$m_5(x, \rho_2, \rho_{12}, \rho_{12}, \rho_1) = x$$
$$\vdots$$

Comparison of the two examples.

First chain complex:



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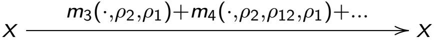

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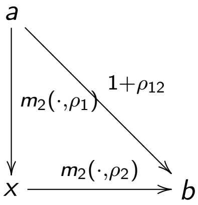

Second chain complex:

$$x \xrightarrow{m_3(\cdot,\rho_2,\rho_1)+m_4(\cdot,\rho_2,\rho_{12},\rho_1)+\dots} x$$

They're A_∞ homotopy equivalent (exercise).

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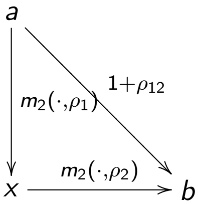

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They're A_∞ homotopy equivalent (exercise).
Suggestive remark:

$$(1+\rho_{12})^{-1} \text{``=''} 1+\rho_{12}+\rho_{12},\rho_{12}+\dots$$
$$\rho_2(1+\rho_{12})^{-1}\rho_1 \text{``=''} \rho_2,\rho_1+\rho_2,\rho_{12},\rho_1+\dots.$$

Again, that's a relief, since:

Theorem

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $(\Sigma, \boldsymbol{\alpha}', \beta', z')$ are pointed bordered Heegaard diagrams for the same bordered Y^3 then $\widehat{\mathrm{CFA}}(\Sigma)$ is A_∞-homotopy equivalent to $\widehat{\mathrm{CFA}}(\Sigma')$.

The pairing theorem

Recall:

Theorem

If $\partial Y_1 = F = -\partial Y_2$ *then*

$$\widehat{\mathrm{CF}}(Y_1 \cup_\partial Y_2) \simeq \widehat{\mathrm{CFA}}(Y_1)\widetilde{\otimes}_{\mathcal{A}(F)}\widehat{\mathrm{CFD}}(Y_2).$$

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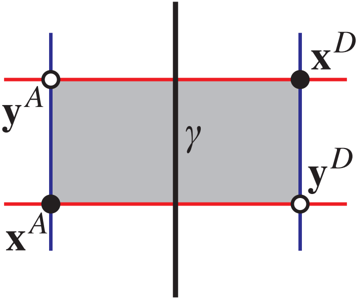
At this point, one might wonder:

- Why the distinction between $\widehat{\mathrm{CFD}}$ and $\widehat{\mathrm{CFA}}$?
- And why is the pairing theorem true?

Consider this local picture



Here,

$$\begin{aligned} d(\mathbf{x}^A \otimes \mathbf{x}^D) &= \mathbf{x}^A \otimes d(\mathbf{x}^D) \\ &= \mathbf{x}^A \otimes \gamma \mathbf{y}^D \\ &= \mathbf{x}^A \gamma \otimes \mathbf{y}^D \\ &= \mathbf{y}^A \otimes \mathbf{y}^D \end{aligned}$$

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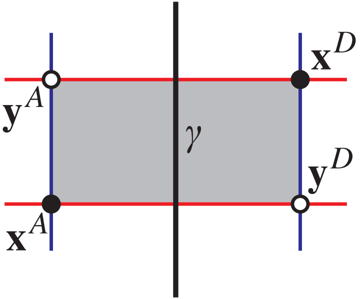

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as desired.

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Computing $\widehat{\text{CFD}}$ for knot complements.

- For a knot K in S^3, $\widehat{\text{CFD}}$ and $\widehat{\text{CFA}}$ are determined by $\text{CFK}^-(K)$.

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- For a knot K in S^3, $\widehat{\mathrm{CFD}}$ and $\widehat{\mathrm{CFA}}$ are determined by $\mathrm{CFK}^-(K)$.
- The proof involves winding one of the α-curves like this
- ...and studying boundary degenerations when curves in a bordered H.D. are allowed to cross z.

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- In particular, one can reprove results of Eaman Eftekhary and Matt Hedden.
- More generally, these techniques imply HFK^- of satellites of K is determined by CFK^- of K. i.e.,

Theorem

Suppose K and K' are knots with $\mathrm{CFK}^-(K)$ filtered homotopy equivalent to $\mathrm{CFK}^-(K')$. Let K_C (resp. K'_C) be the satellite of K (resp. K') with companion C. Then $\mathrm{HFK}^-(K_C) \cong \mathrm{HFK}^-(K'_C)$.