TWO-DIMENSIONAL PERIODIC SCHRÖDINGER OPERATORS AND PRYM'S θ-FUNCTIONS

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Introduction

The general nonrelativistic scalar Schrödinger operator in an external time-independent electromagnetic field F_{ij} has the form

$$\hat{H} = \sum_{\alpha=1}^{n} (\partial_\alpha - ieA_\alpha)^2 + u(x), \quad x = (x^1, \dots, x^n), \tag{1}$$
$$i, j = 0, 1, \dots, n, \quad \alpha = 1, \dots, n, \quad \partial_\alpha = \partial/\partial x^\alpha.$$

By definition, we have electric and magnetic fields:

$$F_{ij} = -F_{ji}, \quad F_{0\alpha} = E_\alpha = \partial_\alpha u, \quad F_{\alpha\beta} = H_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{2}$$

"Gauge" transformations (3) preserve the equation $\hat{H}\psi = \varepsilon\psi$

$$A_\alpha \to A_\alpha + \partial_\alpha \varphi, \quad \psi \to \psi \exp(-ie\varphi), \quad u \to u, \quad \varepsilon \to \varepsilon. \tag{3}$$

Using (3), we may reduce the operator $\hat{H}$ for $n = 1, 2$ to the canonical form:

$$n = 1\colon \hat{H} = \partial^2 + u(x), \quad \partial = \partial/\partial x, \tag{4}$$

$$n = 2\colon \hat{H} = \partial\bar{\partial} + A\bar{\partial} + V(z, \bar{z}), \tag{5}$$
$$z = x + iy, \quad \bar{z} = x - iy, \quad \partial = \partial/\partial z, \quad \bar{\partial} = \partial/\partial \bar{z}.$$

It is well known that remarkable one-parametric families of the linear operators (4) are very important in the soliton theory ([1]):

$$\frac{d\hat{H}}{dt} = [\hat{H}, B_m] \qquad (\text{"Higher KdV's"}). \tag{6}$$

Here $B_m = \partial_x^{2m+1} + \sum_{j=1}^{2m} b_j(u, u_x, \dots)\partial_x^{2m-j}$ is such a linear differential operator, that the equation (6) is equivalent to nonlinear P.D.E.

$$\begin{aligned} &u_t = \Phi(u, u_x, \dots, u_x^{(2m+1)}), \\ &B_1 = \partial_x^3 + \frac{3}{y}(u\partial_x + \partial_x u) \leftrightarrow \Phi_1 = u_{xxx} + 6uu_x, \\ &B_0 = \partial_x \leftrightarrow \Phi_0 = u_x. \end{aligned} \tag{7}$$

Direct generalization of (6) to $n = 2$ is possible *only* in parabolic case (8)

$$\hat{H} = \hat{P} = \sigma\partial_y + \partial_x^2 + u, \quad \sigma \in \mathbb{C}. \tag{8}$$

For example, the well-known KP-equation

$$\begin{aligned} &W_x = u_y, \\ &3\sigma^2 W_y = u_t - 6uu_x - u_{xxx}, \quad \sigma^2 = \pm 1 \end{aligned} \tag{9}$$

is equivalent to “Lax-like” equation:

$$\frac{d\hat{P}}{dt} = [\hat{P}, B], \quad B = \partial_x^3 + \frac{3}{2}u\partial_x + W$$

(V.S. Driuma, A.B. Shabat, V.E. Zakharov 1974).

There is an elementary *theorem*: any Lax-like deformation (6) of the class of all smooth two-dimensional Schrödinger operators (1) is trivial for $n \geqslant 2$.

Nontrivial two-dimensional generalization of the equation (6) was found by S.V. Manakov in [2]:

$$\frac{d\hat{H}}{dt} = [\hat{H}, b] + C\hat{H}. \tag{10}$$

The deformation (10) for a certain linear P.D.O.'s B,C and Schrödinger operator (5) is equivalent to the system of nontrivial, nonlinear P.D.E.'s

$$\begin{aligned} V_t &= \Phi_1(V, A, V_x, A_x, V_y, A_y, \dots), \\ A_t &= \Phi_2(V, A, V_x, A_x, V_y, A_y, \dots) \end{aligned} \tag{11}$$

for all smooth (complex) coefficients $V(x, y, t), A(x, y, t)$.

The equation (10) looks like Lax equation on the set of all solutions of (12):

$$\hat{H}\Psi = 0 \leftrightarrow \left(\frac{d\hat{H}}{dt} - [\hat{H}, B] \right) \Psi = 0. \tag{12}$$

The periodic inverse spectral problem for two-dimensional Schrödinger operator (1) based on the spectral data corresponding to one fixed energy level $\varepsilon = \varepsilon_0$, was posed and considered by B.A. Dubrovin, I.M. Krichver and S.P. Novikov in [3], [4]. It was solved in [3] for a certain class of “algebraic” operators the two-dimensional analog of the well known “finite-zone” operators on the given level $\varepsilon = \varepsilon_0$—see §1. Some nontrivial sufficient “reality” conditions (such that $\hat{H}$ is self-adjoint but $A \neq 0$) were noneffectively found by I.V. Cherednic in [5].

Problem. Which spectral data in [3] provide the real “purely potential” operators (13)? The class (13) is most important (see the end of §1):

$$A \equiv 0, \quad \hat{H} = \partial\bar{\partial} + V(x, y), \quad V \in \mathbb{R}. \tag{13}$$

This problem was partialy solved in the recent papers of S.P. Novikov and A.P. Veselov ([6], [7], [8]) in terms of the Riemann surfaces with some group of involutions and corresponding Prim's θ-functions—see §2.

P.G. Grinevitch and R.G. Novikov have recently found the analog of these results for some class of decreasing potentials $V \to 0$, $x^2 + y^2 \to \infty$ using the technique of S.V. Manakov (“nonlocal Riemann problem”—see [9]). But the conjecture of S.P. Novikov (below) is probably untrue for this class; it is probably not dense in the class of all smooth rapidly decreasing potentials.

The deformations (10) preserving a class of the purely potential self-adjoint operators (13) were found and studied in ???. The simplest and important example is

$$
\begin{aligned}
&\frac{d\hat{H}}{dt} = [\hat{H}, B] + f\hat{H}, \\
&\hat{H} = \partial\bar{\partial} + V, \quad B = \mathcal{D} + \bar{\mathcal{D}}, \quad \mathcal{D} = \partial^3 + u\partial, \\
&V_t = (\partial^3 + \bar{\partial}^3)V + \partial(uV) + \bar{\partial}(\bar{u}V), \\
&\bar{\partial}u = 3\partial V, \quad f = \partial u + \bar{\partial}\bar{u}.
\end{aligned} \tag{14}
$$

It is possible to exploit (14) for the effectivization of θ-functional formulas for $\hat{H}$ (and also for the recognition of Prym's θ-functions, like in [9] for the Jacobian varieties. This program was developed recently by I.A. Taimanov. Some results see in §3).

The last §4 contains a difference analog of the previous theory. There is a special class of the difference operators, whose spectral properties are in some natural sense analogous to the properties of continuous purely potential operators (13). The results of §4 were obtained recently by I.M. Krichever. They may be very useful for the proving of the following conjecture 1.

Conjectures (S.P. Novikov)**.** 1) The algebraic (rank $\ell = 1$) operators $\hat{H}$ corresponding to one energy level generate a dense family among all the smooth real, purely potential periodic operators for $n = 2$.

2) All such algebraic operators have the spectral data described in [7] (the analogous problem is not solved for KP either).

3) Formula (37) determines the solutions of the equation (36), $j = 1$, iff $B_{\mu\nu}$ is the Riemann's matrix of some admissible Prym's variety and $\vec{U}_1, \vec{U}_2, \vec{W}$ are the periods of corresponding differential forms. (The constant c and components W_j are the functions of $B_{\mu\nu}, \vec{U}_1, \vec{U}_2$—see the end of §3).

1. Two-Dimensional Algebraic Operators. Spectral Data and Inverse Problem

First recall some definitions.

Definition 1. Two-dimensional P.D.O. L_1 is algebraic iff there are nontrivial P.D.O.'s L_2, L_3, B_{ij} such that (15) is true

$$[L_i, L_j] = B_{ij}L_1, \quad i, j = 1, 2, 3. \tag{15}$$

General properties of algebraic operators:

1. There is a polynomial $P(\lambda, \mu)$ such that

$$L_1\Psi = 0 \Rightarrow P(L_2, L_3)\Psi = 0. \tag{16}$$

2. Common eigenfunction $\Psi(x, y, \lambda, \mu)$

$$L_1\Psi = 0, \quad L_2\Psi = \lambda\Psi, \quad L_3\Psi = \mu\Psi \tag{17}$$

is meromorphic on the Riemann surface Γ; see its analytical properties below:

$$
\begin{aligned}
&P(\lambda, \mu) = 0, \quad (\lambda, \mu) = Q \in \Gamma, \\
&\Psi(x, y, \lambda, \mu) = \Psi(x, y, Q).
\end{aligned} \tag{18}
$$

Definition 2. Rank of an algebraic operator L_1 is the dimension of the space $\Psi(x, y, Q)$ in a "general" point $Q \in \Gamma$.

We shall discuss in this work only the algebraic operator of rank $\ell = 1$. See the general theory $\ell > 1$ in [10].

Suppose that Γ is nonsingular and rank $\ell = 1$. The analytical properties of algebraic Schrödinger operator (1), were described in [7]:

1. The common normalized eigenfunction $\Psi(x, y, Q)$, $\hat{H}\Psi = 0$, $\Psi(0, 0, Q) \equiv 1$, $Q \in \Gamma$ is meromorphic on $\Gamma \setminus (P_1 \cup P_2)$; the points $P_1 \neq P_2 \in \Gamma$ are some "infinite" points with local parameters $k_1^{-1} = w_1$, $k_2^{-1} = w_2$, $K_i(Q) \to \infty$, $Q \to P_i$, $i = 1, 2$.

2. In general Ψ has g different poles $Q_1, \ldots, Q_g$, whose position is independent of (x, y); $g = g(\Gamma)$ is the genus of Γ.

3. Ψ has the asymptotic (19):

$$
\begin{aligned}
&Q \to P_1, \quad \Psi(x, y, Q) = {}_1(x, y)\ell^{k_1 z}\left(1 + \sum_{i \geqslant 1} \eta_i w_1^i\right), \\
&Q \to P_2, \quad \Psi(x, y, Q) = {}_2(x, y)\ell^{k_2 \bar{z}}\left(1 + \sum_{i \geqslant 1} \xi_i w_2^i\right), \\
&\qquad z = x + iy, \quad \bar{z} = x - iy.
\end{aligned} \tag{19}
$$

Definition 3. The set of quantities $(\Gamma, P_1, P_2, k_1, k_2, Q_1, \ldots, Q_g)$ with the properties 1–3, mentioned above are "spectral data" for generic algebraic Schrödinger operator $L = \hat{H}$ of the general form (1) for $n = 2$ and rank $\ell = 1$.

Theorem 1. *Any spectral data* $(\Gamma, P_1, P_2, k_1, k_2, Q_1, \ldots, Q_g)$ *for nonsingular surface* Γ *with the genus* $g(\Gamma) = g$, *generic divisor* $\mathcal{D} = Q_1 + \cdots + Q_g$, *any two points* $P_1 \neq P_2$ *and* k_1, k_2 *local parameters determine the unique function* $\Psi(x, y, Q)$ *and Schrödinger operator* $\hat{H}$ *such that*

$$
\begin{gathered}
c_1 \equiv 1, \quad \hat{H} = \partial\bar{\partial} + A\bar{\partial} + V, \\
\hat{H}\Psi \equiv 0, \quad A = -\partial \ln c_2, \quad V = -\frac{\partial \eta_1}{\partial z}.
\end{gathered}
$$

The coefficients of $\hat{H}$ *are complex, periodic or quasi-periodic* (*with* $2g$ *quasi-periods*) *functions on* (x, y):

$$
\begin{aligned}
V &= \partial\bar{\partial} \ln \theta(U_1 z + U_2 \bar{z} + \zeta_0 + A(P_1)), \\
A &= -\partial \ln \frac{\theta(U_1 z + U_2 \bar{z} + \zeta_0 + A(P_2))}{\theta(U_1 z + U_2 \bar{z} + \zeta_0 + A(P_1))}, \\
\Psi &= \frac{\theta(A(P) + zU_1 + \bar{z}U_2 + \zeta_0)\theta(A(P_1) + \zeta_0)}{\theta(A(P) + \zeta_0)\theta(A(P_1) + zU_1 + \bar{z}U_2 + \zeta_0)} \exp\left(z\left(\int_{P_0}^{P} \Omega_1 - \alpha\right) + \bar{z}\int_{P_1}^{P} \Omega_2\right).
\end{aligned} \tag{20}
$$

Changes of local parameters $w_1 = aw_1' + \ldots$, $w_2 = bw_2' + \ldots$ leads only to the linear transformation

$$
\begin{gathered}
\hat{H} \to \hat{H}' = a^{-1}b^{-1}(\partial'\bar{\partial}' + A'\bar{\partial}' + V'(z', \bar{z}')), \\
z = az', \quad \bar{z} = b\bar{z}', \quad A' = aA(az', b\bar{z}'), \quad V' = abV(az', b\bar{z}').
\end{gathered}
$$

For the self-adjoint operators $\hat{H}$ with periodic coefficients A, V a function Ψ is a Bloch's function corresponding to the: zero-energy level.

Here (a_j, b_j) is canonic basis of $\hat{H}(\Gamma, \mathbb{Z})$, $\hat{\omega}_1, \ldots, \hat{\omega}_g$—the basis of the holomorphic forms on Γ and Ω_1, Ω_2 are the meromorphic forms with the poles only in P_1, P_2

respectively such that

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij};$$

$$\oint_{a_k} \hat{\omega}_j = 2\pi i \delta_{jk}, \quad \oint_{b_k} \hat{\omega}_\nu = \hat{B}_{\mu\nu} = \hat{B}_{\nu\mu}, \quad \oint_{a_k} \Omega_\alpha = 0,$$

$$\Omega_\alpha = -w_\alpha^{-2} dw_\alpha (1 + \text{reg.}), \quad \alpha = 1, 2, \quad U_\alpha^j = \oint_{b_j} \Omega_\alpha,$$

$$\hat{\theta}(\eta_1, \dots, \eta_g) = \hat{\theta}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\eta_1, \dots, \eta_g), \tag{21}$$

$$\hat{\theta}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\eta_1, \dots, \eta_g) = \sum_{N \in \mathbb{Z}^g} \exp\left\{\frac{1}{2}\langle \hat{B}(N + \alpha), N + \alpha\rangle + \langle \eta + 2\pi i \beta, N + \alpha\rangle\right\},$$

$$A \colon \Gamma \to J(\Gamma) \quad \text{is the “Abel map”:}$$

$$A(P)^i = \int_{P_0}^P \hat{\omega}_i \quad (i = 1, \dots, g),$$

α is constant, which determines from the property

$$\left(\int_{P_0}^P \Omega_1 - \alpha\right) = w_1^{-1} + O(w_1).$$

The important problem is: for which spectral data the corresponding operators have $A \equiv \bar{0}$ and V real and smooth. It will be discussed in §2.

There is an important class of two-dimensional operators in the external periodic (or constant) magnetic field $H(\alpha, y)$ and the electric lattice potential $V(x, y)$:

$$\begin{gathered} \hat{H} = \partial\bar{\partial} + A\partial + V, \quad V(x + T_1, y) = V(x, T_2 + y) = V(x, y), \\ H(x, y) = \bar{\partial}A(x, y) = H(x + T_1, y) = H(x, y + T_2). \end{gathered} \tag{22}$$

For the operators (22) we have periodic fields, but nonperiodic operators. This class is not contained in our theory. Its mathematical theory is quite different—see [11]. Our theory considers only the case in which the average magnetic field $\bar{H}$ (or the magnetic “flux”) is trivial—“topologically trivial” magnetic fields as the cohomology classes on the torus T^2. In this case “physical” magnetic fields are usually identically zero in the real crystals. So the most important case in our theory is $A \equiv 0$ (§2).

2. Schrödinger Operators with the Zero Magnetic Field. Prym's θ-Functions

Simplest examples of the algebraic purely potential operators are (23)

$$\hat{H} = \partial\bar{\partial} + V(x, y), \quad V(x, y) = V_1(x) + V_2(y) \tag{23}$$

(here the operators $H_1 = \partial_x^2 + V_1(x)$ and $H_2 = \partial_y^2 + V(y)$ are “finite-zoned” or “finite-gap” 1-dimensional operators). The operators (23) are algebraic, corresponding to *any* level the last property of (23) makes an exception—see §3.

Theorem 2. 1) *Any spectral data of the theorem* 1, *satisfying the following conditions* a), b), *give purely potential Schrödinger operators* $\hat{H} = \partial\bar{\partial} + V(x, y)$:

a) *the nonsingular surface* Γ *has an involution*

$$\sigma \colon \Gamma \to \Gamma, \quad \sigma^2 = 1$$

such that

$$\sigma(P_1) = P_1, \quad \sigma(P_2) = P_2, \quad \sigma(k_\alpha) = -k_\alpha \quad (\alpha = 1, 2) \tag{24}$$

b) *the divisor of poles* $\mathcal{D} = Q_1 + \cdots + Q_g$ *satisfices the relationship*

$$\mathcal{D} + \sigma(\mathcal{D}) \cong K + P_1 + P_2. \tag{25}$$

Here K *is the canonical divisor* (*a divisor of differential forms*) *and* $\cong$ *means the so-called "linear equivalence" of the divisors.*

2) *The potential* V *is real if specral data have the following properties*:

c) *There is an anti-involution* τ

$$\tau \colon \Gamma \to \Gamma$$

such that the pair (σ, τ) *generates the group* $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$\tau^2 = 1, \quad \tau\sigma = \sigma\tau, \quad \tau(P_1) = P_2, \quad \tau(k_1) = \bar{k}_2 \tag{26}$$

and the divisor $\mathcal{D}$ *is* τ*-invariant*:

$$\tau(\mathcal{D}) = \mathcal{D} \tag{27}$$

Remark. I. Shafarevitch and V. Shockurov explained to us that (25) is solvable iff the involution σ has exactly 2 fixed points P_1, P_2—see ???.

Choose the canonical basis (21) $a_j, b_j \in H_1(\Gamma)$, $j = 1, \dots, g = 2g_0$ the basis of holomorphic differential 1-forms $\hat{\omega}_j$, and the meromorphic differentials Ω_α, $\alpha = 1, 2$ with the properties (23), (28):

$$\sigma(a_i) = a_{i+g_0}, \quad \sigma(b_i) = b_{i+g_0}, \quad i = 1, \dots, g_0. \tag{28}$$

Definition 4. The Prym differentials ω are meromorphic differentials on Γ such that

$$\sigma^*\omega = -\omega.$$

We can construct the basis of the holomorphic Prym differentials from (28)

$$\begin{gathered} \omega_1, \dots, \omega_{g_0}, \quad \omega_i = \hat{\omega}_i - \hat{\omega}_{i+g_0}, \\ \oint_{a_k} \omega_j = \delta_{kj}, \quad B_{kj} = \oint_{b_k} \omega_j = B_{jk}. \end{gathered} \tag{29}$$

The lattice (29) determines some abelian variety $P(\Gamma, \sigma)$ ("Prym variety") and the θ-functions (30), which depend on g_0 variables: $\theta(\eta_1, \dots, \eta_{g_0}) = \theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](\eta_1, \dots, \eta_{g_0})$,

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\eta_1, \dots, \eta_{g_0}) = \sum_{N \in \mathbb{Z}^{g_0}} \exp\left\{\frac{1}{2}\langle B(N+\alpha), N+\alpha\rangle + \langle \eta + 2\pi i\beta, N+\alpha\rangle\right\} \tag{30}$$

Both the meromorphic differentials Ω_α (see theorem 2) are the Prym differentials

$$\sigma^*\Omega_\alpha = -\Omega_\alpha.$$

Any meromorphic differential form $\Omega_\alpha^{(k)}$ which has only one pole P_α and property (31) is the Prym differential:

$$\begin{gathered} \oint_{a_j} \Omega_\alpha^{(k)} = 0, \quad j = 1, \dots, 2g_0, \\ \Omega_\alpha^{(k)} = w_\alpha^{2k} dw_\alpha (1 + \text{reg.}), \quad \sigma^*\Omega_\alpha^{(k)} = -\Omega_\alpha^{(k)}, \\ \Omega_\alpha^{(1)} = \Omega_\alpha, \quad \sigma^* w_\alpha = -w_\alpha. \end{gathered} \tag{31}$$

We have a collection of g_0-vectors $\vec{U}_\alpha^{(k)}$

$$
U_{\alpha j}^{(k)} = \oint_{b_j} \Omega_\alpha^{(k)}, \quad \alpha = 1,2;\ k = 1,2,\dots,\ j = 1,2,\dots,g_0, \qquad (32)
$$
$$
\vec{U}_\alpha^{(1)} = \vec{U}_\alpha.
$$

Theorem 3. *Coefficients and eigenfunctions of the Schrödinger operators from theorem 2 may be written by the following formulas in Prym's θ-functions*:
(33)

$$
V(x,y) = 2\partial\bar\partial \ln \theta(\vec{U}_1 z + \vec{U}_2 \bar z + \vec\zeta_0) + c(\Gamma,\sigma),
$$
$$
\Psi(x,y,P) = \frac{\theta(\eta(P) + zU_1 + \bar z U_2 + \zeta_0)\theta(\zeta_0)}{\theta(\eta(P)+\zeta_0)\theta(zU_1 + \bar z U_2 + \zeta_0)} \exp\left[z\left(\int_{P_0}^{P} \Omega_1 - \alpha\right) + \bar z \int_{P_1}^{P} \Omega_2 \right],
$$
$$
\eta(P)^i = \int_{P_1}^{P} \omega_i \quad (i = 1,\dots,g_0), \quad \hat H \Psi = 0.
$$

The constant g_0-vector ζ_0 depends only on the divisor $\mathcal{D}$. For the real potentials $V(x,y)$ we have a factor-surface Γ_0 with the anti-involution τ_0 induced by τ:

$$
\Gamma_0 = \Gamma/\sigma, \quad \tau_0\colon \Gamma_0 \to \Gamma_0, \quad \tau_0^2 = 1, \quad \tau_0(P_1) = P_2.
$$

The genus $g(\Gamma_0)$ is equal to $g_0 = g/2$. In the general case the anti-involution τ_0 has q smooth fixed ovals S_i:

$$
S_1,\dots,S_q \subset \Gamma_0, \quad \tau|_{S_j} \equiv 1, \quad j = 1,\dots,q \leqslant g_0 + 1, \quad S_i \cap S_j = \varnothing.
$$

By definition, the so-called "M-curves" (Γ_0,τ_0) have exactly the maximal possible number of ovals, $q = g_0 + 1$.

Conjecture. Formula (33) gives the real smooth algebraic potential $V(x,y)$ only if θ is the θ-function of some Prym variety $P(\Gamma,\sigma)$; $\vec{U}_1, \vec{U}_2$—the vectors of the b-periods of the corresponding meromorphic Prym differentials (31) and ζ_0—some admissible constant vector; the set of all admissible constant vectors $P^0_{\mathbb{R}}(\Gamma,\sigma) \subset P(\Gamma,\sigma)$ is always connected and non-empty iff (Γ_0,τ_0) is the M-curve. The corresponding operators are positive only if the conditions of theorem 4, pt. 2, are satisfied.

Theorem 4. 1). *If $Q \in \Gamma$ is such that $\sigma\tau(Q) = Q$ the Bloch's function $\Psi(x,y,Q)$ is bounded for all real $x,y \in \mathbb{R}^2$*

$$
|\Psi(x,y,Q)| < \text{const} < \infty, \quad \sigma\tau(Q) = Q \qquad (34)
$$

(*the fixed ovals of anti-involution $\sigma\tau$ give a "real Fermi-curve" on the level $\varepsilon = 0$*).

2) *Suppose that the pair (Γ_0,τ_0) is M-curve, $\sigma\tau$ has exactly $2g_0+1$ ovals $(a_1',\dots,a_{g_0}',a_1'',\dots,a_{g_0}'',b)$ such that for $\mathcal{D} = Q_1 + \dots + Q_g$, $g = 2g_0$, we have*

$$
\sigma\tau(a_j') = a_j'', \quad \sigma\tau(b) = b, \quad Q_j \in a_j', \quad Q_{g_0+j} \in a_j'', \quad j = 1,\dots,g_0. \qquad (35)
$$

In this case the operator $\hat H = \partial\bar\partial + V$ is positive $\hat H > 0$.

Conjecture. Suppose, that $\sigma\tau$ has no fixed points and τ has exactly $d + 2\ell$ ovals $(b_1,\dots,b_d,a_1',a_1'',\dots,a_\ell',a_\ell'')$ such that $\sigma\tau(b_j') = b_j$, $\sigma\tau(a_q') = a_q''$. The number of different dispersion relations $\varepsilon_j(p_1,p_2)$ (less than zero) is at least S:

$$
d \equiv 1 \pmod 2, \quad \varepsilon_j(p_1,p_2) < 0,
$$
$$
S \geqslant (d-1)/2, \quad j = 1,\dots,S.
$$

Of special interest is the degenerate case in theorem 4, pt. 2. Suppose that we have a family of data $(\Gamma(\lambda), \dots)$ such that:

$$\Gamma(\lambda) \to \Gamma(\lambda_0) = \bar{\Gamma}, \quad \lambda \to \lambda_0,$$
$$b \to \text{point } Q_0 \in \bar{\Gamma}.$$

In this case we obtain so-called "ground state" $\varepsilon = 0$

$$(\hat{H}\varphi, \varphi) > 0, \quad \varphi \in \mathcal{L}_2(\mathbb{R}^2), \quad \hat{H}\Psi(x, y, Q_0) = 0.$$

The Prym's variety of limiting singular curve $\bar{\Gamma}$ with an involution is nonsingular; the corresponding Prym's θ-functions are also nonsingular. Adequate formulas for the ground-state eigenfunction Ψ may be found in [6].

3. Nonlinear Equations as the Deformations of Two-Dimensional Schrödinger Operator

General Schrödinger operator (1) for $n = 2$ has a number of deformations (10). The first examples were found in [6], [8]. The "hierarchy" of all such deformations with multiparametric ψ-function may be easily deduced from:

The function $\psi = \psi(x, y, t_2', t_2'', \dots, t_i', t_i'')$ has the analytical properties like in §1, but the pt. 1 is replaced by $1'$:

$1'$. ψ has the asymptotic $(14')$

$$Q \to P_1, \quad \psi = C_1(x, y)\ell^{k_1 z + \sum_{i \geqslant 2} k_1^i t_i'}\Big(1 + \sum_{i \geqslant 1} \eta_i w_1^i\Big),$$
$$(14') \qquad Q \to P_2, \quad \psi = C_2(x, y)\ell^{k_2 z + \sum_{i \geqslant 2} k_2^i t_i''}\Big(1 + \sum_{i \geqslant 1} \xi_i w_2^i\Big).$$

General formulas for $A(x, y, t', t'')$ and $V(x, y, t', t'')$ may be obtained trivially from (20) by putting additional terms in the argument of (20); these terms are linearly dependent on all t_i', t_i''.

The deformations of purely potential operators were first considered in ???:

Theorem 5. *Any deformation* $(14')$ *such that* $t_{2i}' = t_{2i}'' = 0$ *preserves the class of purely potential operators* ($C_1 = 1$, C_2 *constant*). *The deformation preserves the "reality" property if* $t_{2j}' = t_{2j+1}'' \in \mathbb{R}$. *The latter deformations have the form*

$$\frac{\partial \hat{H}}{\partial t_{2j+1}} = [\bar{H}, a_j \mathcal{D}_j + \bar{a}_j \bar{\mathcal{D}}_j] + C_j \hat{H}, \quad a_j t_{2j+1} = t_{2j+1}' = \bar{t}_{2j+1}'',$$
$$(36) \qquad \hat{H} = \partial\bar{\partial} + V, \quad \mathcal{D}_j = \partial^{2j+1} + u_1^{(j)}\partial^{2j-1} + \dots, \quad a_j \in C,$$
$$\mathcal{D}_1 = \partial^3 + u_1\partial, \quad C_1 = a_1\partial u_1 + \bar{a}_1\bar{\partial}u_1,$$
$$\mathcal{D}_0 = \partial, \quad C_0 = 0.$$

According to the natural variant of the so-called "Novikov conjecture" formula (37) *satisfies* (36) *for* $j = 1$ *iff it corresponds to some pair* (Γ, σ) (*it corresponds to some triple* (Γ, σ, τ) *in the real case—see also* §2):

$$V(x, y, t) = 2\partial\bar{\partial}\ln\theta(U_1 z + U_2\bar{z} + Wt + \xi_0) + c,$$
$$(37) \qquad c = \text{const}, \quad a_1 = 1, \quad t_1 = t, \quad W = U_1^{(2)} + U_2^{(2)}.$$

Definition 5. We call $g_0 \times g_0$ matrix $B_{\mu\nu}$ "generic" if the rank of the matrix

$$(\tilde{\theta}_{11}[n], \tilde{\theta}_{12}[n], \ldots, \tilde{\theta}_{g_0 g_0}[n], \tilde{\theta}[n])$$

is equal to $\frac{g_0(g_0+1)}{2} + 1$.

Here $\tilde{\theta}_{ij}[n] = \partial_i \partial_j \theta\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right](w)|_{w=0}$, $n \in \mathbb{Z}_2^{g_0}$ and $\tilde{\theta}$ is θ-function corresponding to the Riemann matrix $2B$.

Theorem 6 (I.A. Taimanov). *Suppose that matrix $B_{\mu\nu}$ is generic and the g_0-vectors U_1, U_2 are linearly independent. If formula* (37) *satisfies the equation* (36) *for $j = 1$, then vector W and constant c may by calculated as the functions of $U_1, U_2, B_{\mu\nu}$. For $g_0 = 2$ any generic matrix $B_{\mu\nu}$ and independent vectors U_1, U_2 give the algebraic purely potential Schrödinger operator and the solution of* (36) *for $j = 1$, $a_j = 1$, using the formulas for c, W.*

The structure of exact formulas for $c(U_1, U_2, B_{\mu\nu})$ contains very interesting information on some identifies between the θ-constants.

4. Two-Dimensional Periodic Difference Operators

In this paragraph we shall consider the difference analog of the two-dimensional Schrödinger operator

$$L\Psi_{n,m} = a_{nm}\Psi_{n+1,m} + a_{n-1,m}\Psi_{n-1,m} + b_{nm}\Psi_{n,m+1} + b_{n,m-1}\Psi_{n,m-1}. \tag{38}$$

In one-dimensional case such "symmetric" version of the difference Schrödinger operator has been used in [??] for the integration of the difference KdV equation.

Consideration of the difference operators allows us to obtain not only the difference analog of the results presented in the previous paragraphs of this work, but, which is still more important, to construct for these operators the direct algebraic transformation which is connected to one energy level. This means the construction of the Riemann surface and the other data on it, which are the starting point for the solution of the inverse problem.

These investigations have been for the first time made in for the non-selfadjoint operator

$$L\Psi_{nm} = \Psi_{n+1,m+1} + a_{nm}\Psi_{n+1,m} + b_{nm}\Psi_{n,m+1} + c_{nm}\Psi_{nm}. \tag{39}$$

It has been shown, that the operators, which may be naturaly called "finite-gap on the one energy level", have non-zero co-dimension (growing with the periods of the operators) in the space of all periodic operators. It was shown, that eigenfunction of the generic operators has some unexpected analytical properties. The continual limit of such operators has not been yet clarified.

These results do not contradict, of course, the conjecture, which was formulated in [6] and contained in §1, because the continual limit of (39) includes the operators H (1) with non-zero magnetic field.

Operator (38) seem to reflect adequately the principal properties of the purely potential Schrödinger operators. It will be shown below, that all the periodic operators of the form (38) are in some sense "the finite-gap" on the zero energy level.

Let's define, as usual, the variety of the Bloch's functions for the operator L of the form (38) with the periodic coefficients

$$a_{n+2N,m} = a_{n,m+2M} = a_{n,m}; \quad b_{nm} = b_{n+2N,m} = b_{n,m+2M}. \tag{40}$$

Consider the finite-dimensional linear operator $L(w_1, w_2)$, which is the restriction of the operator L on the space of the eigenfunctions of the monodromy operators:

$$\Psi_{n+2N,m} = w_1\Psi_{n,m}, \quad \Psi_{n,m+2M} = w_2\Psi_{n,m}. \tag{41}$$

Bloch's eigenfunctions of the operator Ψ are the meromorphic functions on the variety M^2, which is determined in ???? by the equation

$$Q(E, w_1, w_2) = \det(L(w_1, w_2) - E\ 1) = 0. \tag{42}$$

Below we shall consider on this variety only the complex algebraic curve corresponding to the zero energy level (furthermore, for our construction we need only "half" of this curve).

The zero energy level of the operator L is fixed by the following property. Let's denote by $\Phi^{\pm}$ the subspace of such functions, that $\Psi_{n,m} = 0$ if the difference $n-m$ is odd (even). The operator L maps these subspaces such that

$$L\colon \Phi^{\pm} \to \Phi^{\mp}. \tag{43}$$

That's why the curve $\Gamma_0 \subset M^2$ of the Bloch's solutions of the

$$L\Psi_{n,m} = 0 \tag{44}$$

is the union of two curves $\Gamma_0^{\pm}$ corresponding to the decomposition of the polynomial

$$D(w_1, w_2) = \det L(w_1, w_2) = Q^+(w_1, w_2)Q^-(w_1, w_2) = 0.$$

To each point of the curves $\Gamma_0^{\pm}$ correspond the Bloch's solutions of the (44) which belong to the $\Phi^{\pm}$. The function $\Psi^{\pm}_{n,m}(P)$, $P \in \Gamma_0^{\pm}$ will be meromorphic on $\Gamma_0^{\pm}$ for all n, m if we normalize these solutions by the conditions $\Psi^+_{0,0} \equiv 1$ and $\Psi^-_{0,1} \equiv 1$.

Let's set for each operator L of the form (38) with periodic coefficients, the following "algebro-geometric" data:

$$L \to (\Gamma_0^+, \mathcal{D}^+ = \{Q_1, \dots, Q_g\}). \tag{45}$$

Here $\mathcal{D}^+$ is the divisor of the poles Q_s of the functions $\Psi^+_{n,m}$ on Γ_0^+, which differ from the "infinite" points $P^{\varepsilon_1\varepsilon_2} \in \Gamma_0^+$, $\varepsilon_i = \pm 1$. The points $P^{\varepsilon_1\varepsilon_2}$ are the poles of the functions $w_i^{\varepsilon_i}$.

There exist two types of transformations of the operator which preserve the data (45). They are defined by functions $g^+_{n,m} \in H^{\pm}$. The first transformation corresponds to the multiplication of each equation (44) (i.e. for each pair n, m with odd difference) by the $g^-_{n,m}$:

$$\begin{aligned} &\text{if } n-m \equiv 1 \pmod 2, \text{ then} \quad a_{n,m} \to a_{n,m}g^-_{n,m},\ b_{n,m} \to b_{n,m}g^-_{n,m} \\ &\text{if } n-m \equiv 0 \pmod 2, \text{ then} \quad a_{n,m} \to a_{n,m}g^-_{n+1,m},\ b_{n,m} \to b_{n,m}g^-_{n,m+1} \end{aligned} \tag{46}$$

The second transformation corresponds to the multiplication $\Psi^+_{n,m} \to g^+_{n,m}\Psi^+_{n,m}$:

$$\begin{aligned} &\text{if } n-m \text{ is even, then} \quad a_{n,m} \to a_{n,m}g^+_{n,m}, \quad b_{n,m} \to b_{n,m}g^+_{n,m} \\ &\text{if } n-m \text{ is odd, then} \quad a_{n,m} \to a_{n,m}g^+_{n,m+1}, \quad b_{n,m} \to b_{n,m}g^+_{n+1,m} \end{aligned} \tag{47}$$

Consider the inverse transformation, which reconstructs the operator from the data (45). The operator L corresponding to these data is unique up to the accuracy of transformations (46), (47). As usual, this construction gives exact formulae for Bloch's functions $\Psi^+_{n,m}$.

Let Γ be the algebraic curve of the genus g with fixed points $P^{\varepsilon_1,\varepsilon_2}$, $\varepsilon_i = \pm 1$, $i = 1, 2$. Consider the function $\Psi_{n,m}(P)$, which is meromorphic on Γ. The poles of this function on $\Gamma \setminus P^{\varepsilon_1,\varepsilon_2}$ are $Q_1,\dots,Q_g$. In the neighbourhood of the point $P^{\varepsilon_1,\varepsilon_2}$ the function

$$\Psi_{n,m}(P)k^{\frac{-\varepsilon_1 n-\varepsilon_2 m}{2}} \tag{48}$$

is regular, where $k^{-1}(P)=k^{-1}_{\varepsilon_1,\varepsilon_2}(P)$ is the local parameter in this neighbourhood.

According to the Riemann–Roch theorem the dimension of the linear space of such functions for arbitrary set $Q_1,\dots,Q_g$ in general position equals one.

Lemma 1. *Any function with analytical properties outlined above has the following representation $\Psi_{n,m}(P)=g^+_{n,m}\tilde\Psi_{n,m}(P)$, where $\tilde\Psi_{n,m}(P)$ is given by the formula:*

$$\tilde\Psi_{n,m}=\exp\left(\frac{n+m}{2}\int_{P_0}^{P}\Omega_1+\frac{n-m}{2}\int_{P_0}^{P}\Omega_2\right)\frac{\theta(A(P)+U_1n+U_2m+Z)}{\theta(A(P)+Z)}. \tag{49}$$

Here Ω_1 is the normalized differential on Γ of the third kind with the simple poles in the points $P^{1,1}$ and $P^{-1,-1}$. The residues in this points are equal to $+1$ or -1 respectively. The differential Ω_2 is of the same type but has the poles in the points $P^{+1,-1}$ and $P^{-1,+1}$. The components of the vectors U_1 and U_2 are equal to $U_{1k}=\frac12\oint_{b_k}(\Omega_1+\Omega_2)$, $U_{2k}=\frac12\oint_{b_k}(\Omega_1-\Omega_2)$.

The vector Z equals (after the shift on the vector of the Riemann constants) the image of the divisor $\mathcal{D}^+=\{Q_1,\dots,Q_g\}$ under the Abel's transformation.

Theorem 7. *Let be $\Psi_{n,m}(P)$ be the same as in the previous lemma. Then there exists such an operator L, that the equation (44) is valid. This operator is unique up to accuracy of transformations of the (46) type and its coefficients are produced by the transformation (47) from the coefficients of the operator (which corresponds to the $\tilde\Psi_{n,m}(P)$)*

(50)

$$\begin{aligned}&\tilde a_{nm}=\theta^{-1}(A(P^{1,1})+U_1(n+1)+U_2(m)+\zeta_0),\\ &\tilde b_{nm}=-\theta^{-1}(A(P^{1,1})+U_1n+U_2(m+1)+\zeta_0)\end{aligned}\qquad n-m\equiv 1 \pmod 2,$$

$$\tilde a_{nm}=\frac{\theta(A(P^{-1,1})+U_1(n+1)+U_2(m+1)+\zeta_0)}{\theta(A(P^{1,1})+U_1(n+1)+\zeta_0)\theta(A(P^{-1,1})+U_1n+U_2m+\zeta_0)},$$

$$\tilde b_{nm}=\frac{\theta(A(P^{1,-1})+U_1(n+1)+U_2(m+1)+\zeta_0)}{\theta(A(P^{1,1})+U_1(n+1)+U_2(m+1)+\zeta_0)\theta(A(P^{1,-1})+U_1n+U_2m+\zeta_0)},$$

$$n-m\equiv 0 \pmod 2,$$

If there exists such an anti-involution τ of Γ, that the points $P^{\varepsilon_1,\varepsilon_2}$ are stationary and $\tau(\mathcal{D})=\mathcal{D}$, then the coefficients of L are real. If Γ is the M-curve with fixed ovals $a_1,\dots,a_{g+1}$ and the points Q_s belong to a_s and $P^{\varepsilon_1,\varepsilon_2}$ belong to a_{g+1} then the coefficients of the L have no singularities.

In the general case the coefficients of L are quasiperiodic functions as it follows from (50).

Theorem 8. *The operator L, which is given by the theorem 7, is periodic iff the curve Γ is determined by the following equation:*

$$Q(w_1,w_2)=w_1^M+c_1w_1^{-M}+c_2w_2^N+c_3w_2^{-N}+\sum_{N|i|+M|j|<MN}a_{ij}w_1^iw_2^j=0, \tag{51}$$

and the points $P^{\varepsilon_1,\varepsilon_2}$ are those four points, which comactify affine curve (51). This construction provides all generic operators of the form (38) with the periods $2N,2M$ if N and M are relatively prime.

The proof of the latter statement follows from the comparision of the number of parameters of this construction and the number of the periodic operators (38). As it was mentioned above we consider those operators up to the accuracy of transformations (46), (47). Recall that parameters of construction are coefficients a_{ij}, c_1, c_2, c_3 in the equation (51) and the points Q_s. The number of those points equals the genus of the curve Γ which in its turn in the general position equals MN.

In conclusion we must mention that the variety of the Bloch's functions of L is invariant under the involution

$$\sigma\colon (w_1, w_2, E) \to (w_1^{-1}, w_2^{-1}, E).$$

This involution may be naturaly constrained on each curve Γ_ε corresponding to the fixed energy level $E = \varepsilon$. (In the continual limit in which the points $P^{1,1}, P^{-1,-1}$ and also $P^{1,-1}, P^{-1,1}$ coincide, this involution a will evolve into the involution with the properties, which were described in §2).

The involution σ at the zero energy level transforms the components of Γ into one another

$$\sigma_0\colon \Gamma_0^{\pm} \to \Gamma_0^{\mp}.$$

Unfortunately, we have not yet obtained any effective construction which would allow us to reconstruct $\Psi^-_{n,m}(P)$ from the data (45) (the existence of such construction follows from the previous results).

The operators for which the corresponding curve Γ^+ is invariant for the involution σ (i.e. the polynomial (51) is invariant under the transformation $(w_1, w_2) \to (w_1^{-1}, w_2^{-1})$) and the divisor $\mathcal{D}^+$ satisfies the condition

$$\mathcal{D}^+ + \sigma(\mathcal{D}^+) \simeq K + P^{1,1} + P^{1,-1}$$

have very interesting properties. In the next paper we shall consider this class of the operators in detail.

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