Uhlenbeck compactification as a Bridgeland moduli space

Tuomas Tajakka

University of Washington

February 12, 2021

Outline

1. Dimension 1
   - Slope stability on a curve
   - Projectivity via a determinantal line bundle

2. Dimension 2
   - Gieseker and Uhlenbeck moduli spaces
   - Bridgeland stability
   - Projectivity on vertical wall & Uhlenbeck
   - Proof sketch & questions

3. Dimension 3
   - Work in progress: PT-stability

Dim 1: Slope stability on a curve

- C a smooth, projective curve over $\mathbb{C}$
- (everything will be over $\mathbb{C}$)
- **Slope** of $E \in \mathrm{Coh}(C)$:

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$$

  if $\mathrm{rk}(E) > 0$, and ∞ if $\mathrm{rk}(E) = 0$.
- $E \in \mathrm{Coh}(C)$ is **stable** (resp. **semistable**) if for all subsheaves $F \subsetneq E$

$$\mu(F) < \mu(E) \quad (\text{resp. } \mu(F) \leq \mu(E))$$

- $E \in \mathrm{Coh}(C)$ is **polystable** if

$$E \cong \oplus E_i$$

  where E_i are stable, $\mu(E_i) = \mu(E)$.

- A *semistable* $E \in \mathrm{Coh}(C)$ has **JH-filtration**

$$0 \subseteq E_1 \subseteq \cdots \subseteq E_{n-1} \subseteq E_n = E$$

  where E_i/E_{i-1} are *stable* with $\mu = \mu(E)$.
- Semistable E and E' are **S-equivalent** if they have isomorphic JH-factors.
  S-equivalence classes $\leftrightarrow$ polystable sheaves

$$\begin{array}{cc} \mathcal{M}^{\mu}_{r,d} & \text{moduli stack} \\ \downarrow & \\ M^{\mu}_{r,d} & \text{good moduli space} \end{array}$$

- **Good moduli space**: parameterizes S-equivalence classes $\leftrightarrow$ polystable sheaves of rank r, degree d.
- Stack-theoretic techniques $\Rightarrow$ $M^{\mu}_{r,d}$ exists as a proper algebraic space.

Question: Is $M^{\mu}_{r,d}$ projective?

(Mumford: Yes, using GIT.)

- Faltings: Ample determinantal line bundle.
- $\mathcal{E}$ - universal bundle on $\mathcal{M}^{\mu}_{r,d} \times C$

$$\begin{tikzcd} & \mathcal{E} \otimes q^*(F) \arrow[d, dotted, no head] & & \\ & \mathcal{M}^{\mu}_{r,d} \times C \arrow[dl, "p"'] \arrow[dr, "q"] & F \arrow[d, dotted, no head] \\ \mathcal{M}^{\mu}_{r,d} & & C \end{tikzcd}$$

- For $F \in \mathrm{Coh}(C)$ locally free, set

$$\lambda_{\mathcal{E}}(F) = \det(Rp_*(\mathcal{E} \otimes q^* F))^{\vee}.$$

- *Depends only on* $\mathrm{rk}(F), \det(F)$.
- Fix $\mathrm{rk}(F), \det(F)$ so that

$$\chi(C, E \otimes F) = 0 \text{ for } E \in \mathcal{M}^{\mu}_{r,d}.$$

- (1) $\lambda_{\mathcal{E}}(F)$ descends to $M^{\mu}_{r,d}$.
  (2) $Rp_*(\mathcal{E} \otimes q^* F)$ is a 2-term complex

$$[\mathcal{G}_0 \xrightarrow{g} \mathcal{G}_1], \quad \mathrm{rk}(\mathcal{G}_0) = \mathrm{rk}(\mathcal{G}_1).$$

  Global section of $\lambda_{\mathcal{E}}(F)$:
  $s_F = \det(g) : \mathcal{O} \to \det(\mathcal{G}_1) \otimes \det(\mathcal{G}_0)^{\vee}$
- Vary F with fixed $\mathrm{rk}(F), \det(F)$ $\Rightarrow$ *vary the section* s_F.
- **Thm** (*Faltings 1*): For $E \in \mathcal{M}^{\mu}_{r,d}$ $\exists$ F s.t.

$$H^0(X, E \otimes F) = H^1(X, E \otimes F) = 0$$

  i.e. $s_F(E) \neq 0 \Rightarrow \lambda_{\mathcal{E}}(F)$ is globally generated.
- **Thm** (*Faltings 2*): $\lambda_{\mathcal{E}}(F)$ is *strictly nef*: $\deg(\lambda_{\mathcal{E}}(F)|_{C'}) > 0$ for any $C' \subseteq M^{\mu}_{r,d}$.

  $\Rightarrow$ Induced map $M^{\mu}_{r,d} \to \mathbb{P}^N$ is finite, so $\lambda_{\mathcal{E}}(F)$ is ample.

X smooth, projective surface, $H \subseteq X$ very ample divisor
$v \in K_{\mathrm{num}}(X)$ class with $\mathrm{rk}(v) > 0$

Gieseker stability

- Define stability using **reduced Hilbert polynomial**

$$p_E(n) = \frac{1}{\mathrm{rk}(E)}\chi(X, E(nH))$$

- Good moduli space M_v^{Gies} exists as a projective variety - constructed using GIT.
- M_v^{Gies} parameterizes polystable sheaves – torsion-free but not necessarily locally free.

$$\begin{tikzcd} \mathcal{M}_v^{\mathrm{Gies}} \arrow[r, hook, "\text{open emb.}"] \arrow[d, "\text{gms}"] & \mathcal{M}_v^{\mu} \arrow[d, "\text{not gms}"] \\ M_v^{\mathrm{Gies}} \arrow[r] & M_v^{\mathrm{Uhl}} \end{tikzcd}$$

μ-stability

- Define stability using *H*-**slope**

$$\mu(E) = \frac{H \cdot c_1(E)}{\mathrm{rk}(E)}$$

- No good moduli space in general, but **Uhlenbeck compactification** comes close:
- μ-semistable sheaves are torsion-free $\Rightarrow \quad 0 \to E \to E^{\vee\vee} \to Q \to 0$ with $\dim(Q) = 0$

Thm (J. Li): Projective scheme M_v^{Uhl} param. μ-polystable sheaves up to:

1. isomorphic $E^{\vee\vee}$
2. equal $l_p(Q)$ for all $p \in X$

Fundamentals

- Bridgeland (2003): generalize stability from $\mathrm{Coh}(X)$ to $D^b(X)$.
- A **stability condition** is $\sigma = (\mathcal{A}, Z)$:
  - $\mathcal{A} \subseteq D^b(X)$ heart of a t-structure (full abelian subcategory)
  - $Z : K_{\mathrm{num}}(X) \to \mathbb{C}$ group hom.
  - $Z(\mathcal{A}) \subset \overline{\mathbb{H}} = \{z \mid 0 < \arg(z) \leq \pi\}$
- $E \in \mathcal{A}$ is **semistable** if for subobjects $F \subseteq E$

$$\arg(Z(F)) \leq \arg(Z(E)).$$

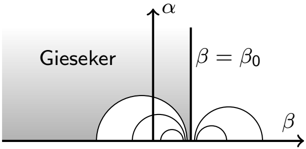
- **Thm** (Bridgeland): Stability conditions parameterized by a *complex manifold* $\mathrm{Stab}(X)$. For fixed $v \in K_{\mathrm{num}}(X)$, wall-and-chamber structure on $\mathrm{Stab}(X)$.
- Existence is nontrivial – $\mathrm{Coh}(X)$ doesn't work as $\mathcal{A}$.

Construction of stability conditions

- Construct $\mathcal{A}$ by *tilting* $\mathrm{Coh}(X)$: write $\mathrm{Coh}(X) = \langle \mathcal{T}, \mathcal{F} \rangle$, set $\mathcal{A} = \langle \mathcal{F}, \mathcal{T}[-1] \rangle$.
- For $\alpha, \beta \in \mathbb{R}, \alpha > 0$, set

$$Z_{\alpha,\beta}(E) = \int_X e^{-(\alpha+\beta i)H}\,\mathrm{ch}(E)$$
$$\mathcal{A}_\beta = \{E \mid F \to E \to T[-1]\}$$

  where $F, T \in \mathrm{Coh}(X), \mu(F) \leq \beta < \mu(T)$.
- **Thm** (B, A–B): $(\mathcal{A}_\beta, Z_{\alpha,\beta}) \in \mathrm{Stab}(X)$.



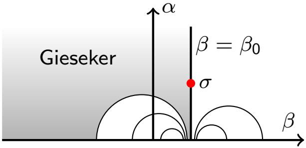
- **Thm** (T, A–HL–H): Moduli stack $\mathcal{M}_v^\sigma$ is algebraic, of finite type, universally closed, good moduli space M_v^σ is a *proper algebraic space*.

Dim 2: Projectivity, vertical wall, and Uhlenbeck

Question: Is M_v^σ projective?

- Previously know cases (incomplete list):
  - $X = \mathbb{P}^2$ (ABCH), $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{Bl}_p\mathbb{P}^2$ (AM) - exceptional collections and quiver GIT.
  - Gieseker stability for any X. (B)
  - $X =$ K3 (BM) or Enriques (N): relate to Gieseker by Fourier-Mukai.
- **Thm** (B–M): (cf. *Faltings 2*) Determinantal line bundle $\mathcal{L}_\sigma$ that is *strictly nef* on M_v^σ:

  $\deg(\mathcal{L}_\sigma|_C) > 0$ for $C \subseteq M_v^\sigma$.
- $\Rightarrow$ If some $\mathcal{L}_\sigma^{\otimes n}$ is globally generated, then $\mathcal{L}_\sigma$ is ample!



New affirmative case: vertical wall

- Polystable objects: $F \oplus (\oplus_i \mathcal{O}_{p_i}[-1])$,
  F μ-polystable *locally free*, and $p_i \in X$
  $\Rightarrow$ set-theoretic bijection with M_v^{Uhl} (L–Q).
- Uhlenbeck-equiv. becomes *S-equiv. for* σ:
  If $E \in \mathrm{Coh}(X)$ is μ-stable but *not locally free*, rotate $0 \to E \to E^{\vee\vee} \to Q \to 0$

$$\begin{aligned} \Rightarrow \quad E &\overset{S}{\sim} E^{\vee\vee} \oplus Q[-1] \\ &\overset{S}{\sim} E^{\vee\vee} \oplus \bigoplus_{p \in X} \mathcal{O}_p^{\oplus l_p(Q)}[-1] \end{aligned}$$

Thm (T): On vertical wall:

$\mathcal{L}_\sigma$ is ample and M_v^σ is projective.

Bijective morphism $M_v^{\mathrm{Uhl}} \to M_v^\sigma$.

$$\begin{tikzcd} \mathcal{M}_v^{\mathrm{Gies}} \arrow[r, hook, "\text{open}"] \arrow[d, "\text{gms}"] & \mathcal{M}_v^{\mu} \arrow[r, hook, "\text{open}"] \arrow[d] & \mathcal{M}_v^{\sigma} \arrow[d, "\text{gms}"'] \\ M_v^{\mathrm{Gies}} \arrow[r] & M_v^{\mathrm{Uhl}} \arrow[r, "\text{bij.}"] & M_v^{\sigma} \end{tikzcd}$$

Dim 2: Proof sketch & questions

Proof sketch

- Special to vertical wall: there are curves $C \in |mH|$ and $F \in \mathrm{Coh}(C)$ s.t. $\mathcal{L}_\sigma^{\otimes n} \cong \lambda_{j^*\mathcal{E}}(F)^\vee = \det(Rp_*(j^*\mathcal{E} \otimes q_C^* F))^\vee$.

$$\begin{tikzcd}
 & \mathcal{E} \arrow[d, dotted, no head] & j^*\mathcal{E} \arrow[d, dotted, no head] \\
 & \mathcal{M}_v^\sigma \times X \arrow[dl, "p"'] \arrow[d] & \mathcal{M}_v^\sigma \times C \arrow[l, hook', "j"] \arrow[d, "q_C"] \\
\mathcal{M}_v^\sigma & X & C \arrow[l, hook']
\end{tikzcd}$$

- Strategy: restriction to C & *Faltings 1*.
- **Lemma 1**: for any $E \in \mathcal{M}_v^\sigma$ and $C \in |mH|$, $H^i(C, F \otimes E|_C) = 0$ for $i \neq 0, 1$,

$$\dim H^0(C, F \otimes E|_C) = \dim H^1(C, F \otimes E|_C)$$

$\Rightarrow \lambda_{j^*\mathcal{E}}(F)^\vee = \det[\mathcal{G}_0 \xrightarrow{g} \mathcal{G}_1]$ has section $s_F = \det(g)$.

- Enough to show: for $[E_0] \in \mathcal{M}_v^\sigma$ there is F with $s_F(E_0) \neq 0$.

- **Thm** (F): If $F \in \mathrm{Coh}(X)$ is μ-semistable, for generic $C \in |mH|$, restriction $F|_C$ is semistable.
- **Lemma 2**: For generic $C \in |mH|$, restriction $E_0|_C$ is a semistable sheaf.
- Apply *Faltings 1*: choose $F \in \mathrm{Coh}(C)$ s.t.

$$H^0(C, F \otimes E_0|_C) = H^1(C, F \otimes E_0|_C) = 0$$

$\Rightarrow s_F(E_0) \neq 0$.

- $M_v^{\mathrm{Uhl}} \to M_v^\sigma$ for free from Li's construction.

Questions

- Is $M_v^{\mathrm{Uhl}} \to M_v^\sigma$ isomorphism?
  (Could have different singularities/non-reduced structure.)
- Projectivity of other Bridgeland moduli spaces? For starters, is $\mathcal{L}_{\sigma_{\mathrm{Gies}}}$ relatively ample for $M_v^{\mathrm{Gies}} \to M_v^\sigma$?
- Projectivity of other moduli of sheaves or complexes when GIT isn't available?

Dim 3: PT-stability (work in progress)

X smooth, projective 3-fold, $H \subseteq X$ very ample

$v \in K_{\mathrm{num}}(X)$ with $\mathrm{rk}(v) > 0$

- A **PT-stable pair** is $\mathcal{O}_X \xrightarrow{f} F$ with F pure dim 1, $\dim(\mathrm{coker}(f)) = 0$.
- (Bayer, Toda) Higher rank PT-pairs from *polynomial/limit stability*.
- Heart of "perverse sheaves" $\mathcal{A}^p \subseteq D^b(X)$ (non-noetherian!), $Z : K_{\mathrm{num}}(X) \to \mathbb{C}[t]$.
- Stability on $\mathcal{A}^p$ using $\arg(Z(E))$ for $t \gg 0$.
- PT-semistable $E \in \mathcal{A}^p$ fits in $F \to E \to T[-1]$, where
  - F is μ-semistable
  - T is 0-dimensional
  - $\mathrm{Hom}(\mathcal{O}_p[-1], E) = 0$ for all $p \in X$
  - $\Rightarrow F^{\vee\vee}/F$ is pure 1-dimensional
- **Thm** (Lo): Moduli stack $\mathcal{M}_v^{\mathrm{PT}}$ is algebraic, of finite type, universally closed.

  If $\gcd(\mathrm{rk}(v), c_1(v)) = 1$, $\mathcal{M}_v^{\mathrm{PT}}$ has a proper coarse moduli space M_v^{PT}.

Thm (T):

A determinantal line bundle $\mathcal{L}_2$ on $\mathcal{M}_v^{\mathrm{PT}}$ is globally generated.

Fibers of induced map $\pi : M_v^{\mathrm{PT}} \to \mathbb{P}^N$:

- $F^{\vee\vee}$ are S-equivalent
- lengths of $F^{\vee\vee}/F$ at codim. 2 points constant
- (incomplete description)

- Global generation follows by restricting to curves, fibers using *Faltings 2*.
- Questions:
  - Does $\mathcal{M}_v^{\mathrm{PT}}$ have a good moduli space when $\gcd \neq 1$? Is it projective?
  - Precise description of fibers of π?
  - Image of π = analog of Uhlenbeck compactification for a 3-fold?