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STACKS

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Section 1. Introduction

Stacks are defined in this document. See [DM69].

Section 2. Definition

Let $\mathcal{C}$ be a site. The 2-category of stacks over $\mathcal{C}$ will be a full sub-2-category of the 2-category of categories over $\mathcal{C}$, see Categories, Definition 3.1.5. Thus a stack will be given by a functor of categories $p : \mathcal{S} \to \mathcal{C}$ which satisfies the following conditions:

(1) $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids, see Categories, Definition 3.1.1,
(2) descent for morphisms holds, and
(3) descent data for objects are effective.

Subsection 2.1. **Explanation.** To explain this, we choose a collection of pullback functors as in Categories, Lemma 3.1.3. Another approach is to use Categories, Lemma 3.3.2.

First, suppose that $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ are objects in the fibre category over U. We are going to define a contravariant functor

$$\mathrm{Isom}(x, y) : \mathcal{C}/U \longrightarrow \mathrm{Sets}.$$

In other words this will be a presheaf on $\mathcal{C}/U$, see Sites, Definition 2.1.5. Namely, for $f : V \to U$ we set

$$\mathrm{Isom}(x, y)(f : V \to U) = \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y).$$

We also have to define the restriction map corresponding to a morphism $(g, \mathrm{id}_U) : (f' : V' \to U) \to (f : V \to U)$ in $\mathcal{C}/U$ (in other words $f' = f \circ g$). This will be a map

$$\mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y) \longrightarrow \mathrm{Mor}_{\mathcal{S}_{V'}}({f'}^*x, {f'}^*y).$$

This map will basically be g^*, except that this transforms an element ϕ of the left hand side into an element $g^*\phi$ of $\mathrm{Mor}_{\mathcal{S}_{V'}}(g^*f^*x, g^*f^*y)$. At this point we use the transformation t of Categories, Lemma 3.1.3. In a formula, the restriction map maps ϕ to

$$t_y \circ g^*\phi \circ (t_x)^{-1}.$$

Lemma 2.1.1. *This actually does give a presheaf.*

Proof. Let $(g', \mathrm{id}_U) : (f' : V' \to U) \to (f : V \to U)$ and $(g'', \mathrm{id}_U) : (f'' : V'' \to U) \to (f' : V' \to U)$ be morphisms in $\mathcal{C}/U$, and let $\phi \in Mor_{\mathcal{S}_V}(f^*x, f^*y)$. It suffices to show that

$$[(\mathrm{Isom}(x,y))(g' \circ g'')](\phi) = [(\mathrm{Isom}(x,y))(g'')]([(\mathrm{Isom}(x,y))(g')](\phi))$$

By Lemma 3.1.3 there are pullback functors $r : (g' \circ g'')^* f'^* \longrightarrow f''^*$, $t' : g'^* f^* \longrightarrow f'^*$, $t'' : g''^* f'^* \longrightarrow f''^*$ and $u : g''^* g'^* \longrightarrow (g' \circ g'')^*$. It follows from the uniqueness part of the lemma that $t''_y \circ g''^* t'_y = r_y \circ u_{f^*y}$ and $g''^* t'^{-1}_x \circ t''^{-1}_x = u^{-1}_{f^*x} \circ r^{-1}_x$. We now have that

$$\begin{aligned}
[(\mathrm{Isom}(x,y))(g' \circ g'')](\phi) &= r_y \circ (g' \circ g'')^* \phi \circ r_x^{-1} \\
&= r_y \circ u_{f^*y} \circ g''^* g'^* \phi \circ u^{-1}_{f^*x} \circ r_x^{-1} \\
&= t''_y \circ g''^* t'_y \circ g''^* g'^* \phi \circ g''^* t'^{-1}_x \circ t''^{-1}_x \\
&= t''_y \circ g''^* (t'_y \circ g'^* \phi \circ t'^{-1}_x) \circ t''^{-1}_x \\
&= [(\mathrm{Isom}(x,y))(g'')]([(\mathrm{Isom}(x,y))(g')](\phi)).
\end{aligned}$$

Alternatively we can argue with Lemma 3.3.2 which says that $\mathcal{S} \to \mathcal{C}$ is equivalent to the category assoicated to a contravariant function $F \colon \mathcal{C} \to \text{Groupoids}$. Then t_y is the identity transformation for all y so the restriction map is $g \mapsto g^*\phi = F(g)\phi$ which is clearly functorial making $\mathrm{Isom}(x,y)$ a presheaf. □

OK, so the second condition listed in Section 2 is simply the condition that $\mathrm{Isom}(x,y)$ is a sheaf on the site $\mathcal{C}/U$!

In order to explain the meaning of the third condition, we must define a descent datum. First, we introduce some notation. Suppose that $\{f_i : U_i \to U\}_{i \in I}$ is a covering in the site $\mathcal{C}$. We will be looking at fibre products $U_{ij} = U_i \times_U U_j$ and $U_{ijk} = U_i \times_U U_j \times_U U_k$. It is very important to allow $i = j$ and even $i = j = k$ in the following. The projection maps from U_{ij} to U_i, resp. U_j are denoted pr_1, resp. pr_2. The projection maps from U_{ijk} to the twofold fibre products are $\mathrm{pr}_{12} : U_{ijk} \to U_{ij}$, $\mathrm{pr}_{13} : U_{ijk} \to U_{ik}$, and $\mathrm{pr}_{23} : U_{ijk} \to U_{jk}$. The projection maps from U_{ijk} to U_i, U_j, and U_k are pr_1, pr_2, and pr_3, respectively.

This notation is potentially ambiguous, but it is standard in the literature. For an example of the ambiguity: note the relations $\mathrm{pr}_1 = \mathrm{pr}_1 \circ \mathrm{pr}_{12} = \mathrm{pr}_1 \circ \mathrm{pr}_{13}$, $\mathrm{pr}_2 = \mathrm{pr}_2 \circ \mathrm{pr}_{12} = \mathrm{pr}_1 \circ \mathrm{pr}_{23}$, and $\mathrm{pr}_3 = \mathrm{pr}_2 \circ \mathrm{pr}_{13} = \mathrm{pr}_3 \circ \mathrm{pr}_{23}$.

Let $x_i \in \mathrm{Ob}(\mathcal{S}_{U_i})$, $i \in I$ be a collection of objects, and let

$$\phi_{ij} : \mathrm{pr}_1^* x_i \longrightarrow \mathrm{pr}_2^* x_j \qquad (i, j \in I)$$

be a collection of morphisms in the fibre categories $\mathcal{S}_{U_{ij}}$. In the definition that follows we will identify, for instance, $\mathrm{pr}_1^* x_i$ on U_{ijk} with both $\mathrm{pr}_{12}^* \mathrm{pr}_1^* x_i$ and $\mathrm{pr}_{13}^* \mathrm{pr}_1^* x_i$.

Definition 2.1.2. The collection $\{x_i, \phi_{ij}\}$ is a descent datum if the following cocycle condition is satisfied: For every triple $(i,j,k) \in I^3$ the diagram

$$\begin{tikzcd}
\mathrm{pr}_1^* x_i \arrow[rr, "\mathrm{pr}_{13}^* \phi_{ik}"] \arrow[dr, "\mathrm{pr}_{12}^* \phi_{ij}"'] & & \mathrm{pr}_3^* x_k \\
& \mathrm{pr}_2^* x_j \arrow[ur, "\mathrm{pr}_{23}^* \phi_{jk}"'] &
\end{tikzcd}$$

in the fibre category $\mathcal{S}_{U_{ijk}}$ is commutative.

Remarks 2.1.3. Two remarks about this definition are in order.

(1) There is a diagonal morphism $\Delta_i : U_i \to U_{ii}$. We can pull back ϕ_{ii} via this morphism to get an automorphism ${\Delta_i}^*\phi_{ii} \in \mathrm{Aut}_{U_i}(x_i)$. On pulling back the cocycle condition for the triple (i, i, i) by $\Delta_{123} : U_i \to U_{iii}$ we deduce that ${\Delta_i}^*\phi_{ii} \circ {\Delta_i}^*\phi_{ii} = {\Delta_i}^*\phi_{ii}$; thus ${\Delta_i}^*\phi_{ii} = \mathrm{id}_{x_i}$.

(2) There is a morphism $\Delta_{13} : U_{ij} \to U_{iji}$ and we can pull back the cocycle condition for the triple (i, j, i) to get the identity $(\sigma^*\phi_{ji}) \circ \phi_{ij} = \mathrm{id}_{\mathrm{pr}^*_{1/2}x_i}$, where $\sigma : U_{ij} \to U_{ji}$ is the switching morphism.

A morphism of descent data $\alpha : \{x_i, \phi_{ij}\} \to \{y_i, \psi_{ij}\}$ is given by a collection of morphisms $\alpha_i : x_i \to y_i$ in the fibre category over U_i such that the following diagrams

$$\begin{tikzcd}
\mathrm{pr}_1^*x_i \arrow[r, "\mathrm{pr}_1^*\alpha_i"] \arrow[d, "\phi_{ij}"'] & \mathrm{pr}_1^*y_i \arrow[d, "\psi_{ij}"] \\
\mathrm{pr}_2^*x_j \arrow[r, "\mathrm{pr}_2^*\alpha_j"'] & \mathrm{pr}_2^*y_j
\end{tikzcd}$$

commute. Note that every morphism of descent data is an isomorphism.

An object $x \in \mathrm{Ob}(\mathcal{S}_U)$ gives rise to a descent datum in the following manner. First we set $x_i = f_i^*x$. Then we set $\phi_{ij} = {t_j}^{-1} \circ t_i$, where $t_i : \mathrm{pr}_1^*f_i^*x \to (f_i \circ \mathrm{pr}_1)^*x$ and $t_j : \mathrm{pr}_2^*f_j^*x \to (f_j \circ \mathrm{pr}_2)^*x$ are the canonical isomorphisms guaranteed by Categories, Lemma 3.1.3. The lemma below shows this is a descent datum; we will call this the *canonical descent datum* associated to x.

Lemma 2.1.4. *The cocycle condition holds for the datum described above.*

Proof. First, note that $f_i \circ \mathrm{pr}_1 = f_j \circ \mathrm{pr}_2 = f_k \circ \mathrm{pr}_3$. Then note that $\mathrm{pr}_{13}^*\phi_{ik}$, $\mathrm{pr}_{23}^*\phi_{jk}$ and $\mathrm{pr}_{12}^*\phi_{ij}$ factor uniquely through $(f_i \circ \mathrm{pr}_1)^*x = (f_j \circ \mathrm{pr}_2)^*x = (f_k \circ \mathrm{pr}_3)^*x$ by Lemma 3.1.3. □

Definition 2.1.5. A descent datum $\{x_i, \phi_{ij}\}$ is said to be effective when there exists an $x \in \mathrm{Ob}(\mathcal{S}_U)$ such that $\{x_i, \phi_{ij}\}$ is isomorphic to the canonical descent datum associated to x.

At this point we are ready to give the definition of a stack.

Definition 2.1.6. A stack (in groupoids) over a site $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ such that

(1) $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids over $\mathcal{C}$,
(2) for all $U \in \mathrm{Ob}(\mathcal{C})$ and all $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ the presheaf $\mathrm{Isom}(x, y)$ is a sheaf on $\mathcal{C}/U$, and
(3) for all coverings $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{C}$, all descent data $\{x_i, \phi_{ij}\}$ for $\mathcal{U}$ are effective.

Usually the hardest part to check is the third condition.

Lemma 2.1.7. *Suppose that $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ are morphisms of stacks over $\mathcal{C}$. Let $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$, p, q, ψ be the explicit 2-fibre product of f and g in the 2-category of categories over $\mathcal{C}$ described in Lemma 3.1.7. Then $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is a stack. In particular the 2-category of stacks over $\mathcal{C}$ has 2-fibre products (and they are as described in Lemma 3.1.7).*

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Proof. FIXME. □

Subsection 2.2. **Examples.** FIXME: Here we need lots of examples.
FIXME: To be continued.

To continue reading,

(1) visit the next section: Stacks and Groupoids, Section 1, or
(2) go back to the table of contents: index.html#contents.

References

[DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Publ. Math. IHES*, 36:75–110, 1969.