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SITES AND SHEAVES

Contents

Section 1. Introduction

The notion of a site was introduced by Grothendieck to be able to study sheaves in the étale topology of schemes. The basic reference for this notion is perhaps [MA71].

Section 2. Topologies

In this section we define what sites are and how to think about sheaves on sites. See [Art62].

Subsection 2.1. **Definitions.** Let C be a category, see Conventions, Section 3. In the following the notation $\{U_i \to U\}_{i\in I}$ means that $U \in \mathrm{Ob}(C)$, that I is a set and that for each $i \in I$ we are given a morphism $U_i \to U$ of C with target U. This notation suggests an open covering as in topology.

Definition 2.1.1. A *site* is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}\mathcal{C}$ consisting of coverings $\{U_i \to U\}_{i\in I}$ satisfying the following axioms

1. If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}\mathcal{C}$.
2. If $\{U_i \to U\} \in \mathrm{Cov}\mathcal{C}$ and for each i we have $\{V_{ij} \to U_i\} \in \mathrm{Cov}\mathcal{C}$, then $\{V_{ij} \to U\} \in \mathrm{Cov}\mathcal{C}$.
3. If $\{U_i \to U\} \in \mathrm{Cov}\mathcal{C}$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\} \in \mathrm{Cov}\mathcal{C}$.

Example 2.1.2. Let X be a topological space. Let T_X be the category whose objects consist of all the open sets U in X and whose morphisms are just the inclusion maps. That is, there is at most one morphism between any two objects in T_X. Now define a site on this category by defining $\{U_i \to U\} \in \mathrm{Cov}T_X$ if $\bigcup U_i = U$. Conditions (1) and (2) above are clear, and (3) is also clear once we realize that in T_X we have $U \times V = U \bigcap V$. Presheaves and sheaves (as defined below) on the site T_X will agree exactly with the usual notion of a (pre)sheaf on a topological space.

Example 2.1.3. Every category (with products) has a canonical topology associated to it. (Note: this is the finest topology where all representable presheaves are sheaves). Here is one example. Let G be a group. Consider the category whose objects are sets X with a left G-action, with G-equivariant maps as the morphisms. We define a topology by declaring $\{U_i \to U\}$ to be a covering if $\bigcup U_i = U$. To verify that fibred products do exist in this category, suppose $f : S \to U$ and $h : T \to U$ are morphisms in the category. Then let $W = \{(s,t) \in S \times T \mid f(s) = h(t)\}$. This is a G-set. The action is given by $g \bullet (s,t) = (g \bullet s, g \bullet t)$. Projections onto S and T are clearly G-maps and so $W = S \times_U T$. Conditions (1), (2), and (3) are now easily verified. This site will be denoted T_G.

Example 2.1.4. FIXME. We can have a lot of examples linked from here.

In this context a presheaf of sets is a contravariant functor $\mathcal{F}$ from $\mathcal{C}$ to Sets (see Categories, Remark 2.1.4). So for every object U of $\mathcal{C}$ we have a set $\mathcal{F}(U)$ and for every morphism $f : V \to U$ a "restricton map" $f^* : \mathcal{F}(U) \to \mathcal{F}(V)$. Sometimes we say an element of $\mathcal{F}(U)$ is a section of $\mathcal{F}$ over U , and sometimes we use the notation $f^*(s) =: s|_V$.

Similarly, we can define the notion of a presheaf of abelian groups, rings, or a presheaf with values in a category. Note also that a presheaf is defined without reference to the topology.

Definition 2.1.5. A presheaf $\mathcal{F}$ on a category $\mathcal{C}$ with values in a category $\mathcal{A}$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{A}$, i.e., $\mathcal{F} : \mathcal{C}^\circ \to \mathcal{A}$.

The sheaf condition is taken with respect to the coverings of the topology as follows.

Definition 2.1.6. Let $\mathcal{F} : \mathcal{C}^\circ \to \mathcal{A}$ be a presheaf with values in a category $\mathcal{A}$. We say that $\mathcal{F}$ is a sheaf if for every covering $\{U_i \to U\} \in \mathrm{Cov}\mathcal{C}$ the diagram

$$(*) \qquad \mathcal{F}(U) \longrightarrow \prod\nolimits_i \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{ij} \mathcal{F}(U_i \times_U U_j)$$

is exact. The meaning of this will be made clear below.

First, we explain what this means if $\mathcal{F}$ is a presheaf of sets: It means that given sections $s_i \in F(U_i)$ such that $\mathrm{pr}_1^*(s_i) = \mathrm{pr}_2^*(s_j)$ in $\mathcal{F}(U_i \times_U U_j)$ for all i, j there exists a unique $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

If $\mathcal{A}$ is arbitrary, the condition means that for any object $X \in \mathrm{Ob}(\mathcal{A})$ the diagram of sets

$$\mathrm{Mor}_\mathcal{A}(X, \mathcal{F}(U)) \longrightarrow \prod\nolimits_i \mathrm{Mor}_\mathcal{A}(X, \mathcal{F}(U_i)) \rightrightarrows \prod\nolimits_{ij} \mathrm{Mor}_\mathcal{A}(X, \mathcal{F}(U_i \times_U U_j))$$

is exact as before. If the products in $(*)$ exist then this condition just means that the first arrow is the equalizer of the other two.

FIXME: A little about the canonical topology, and in particular some examples.

Example 2.1.7. As an example, suppose $\mathcal{F}$ is a sheaf of sets on T_G (see Example 2.1.3). First we note that G itself is an object in the category (the action given by left multiplication). As a G-set, denote it ${}_GG$. Next, remark that the map

$$\mathrm{Hom}_G({}_GG, {}_GG) \longrightarrow G^{opp}, \varphi \longmapsto \varphi(1)$$

is an isomorphism of groups. The inverse map sends $g \in G$ to the map $s \mapsto s \cdot g$ (i.e. right multiplication). Then $\mathcal{F}({}_GG)$ is also a G-set where the action $g \bullet s$ for $g \in G$ and $s \in \mathcal{F}({}_GG)$ is given by $\mathcal{F}(\cdot g)(s)$. Claim: If $\mathcal{F}$ is a sheaf then we can recover $\mathcal{F}$

from the G-set $\mathcal{F}({}_GG)$ and vice versa. That is, there is an equivalence of categories between left G-sets and sheaves of sets on T_G. We will show a quasi-inverse of the functor $\mathcal{F} \mapsto \mathcal{F}({}_GG)$ is given by $U \mapsto \mathrm{Hom}_G(\cdot, U)$ where U is a G-set. Since T_G has the canonical topology, the presheaves $\mathrm{Hom}_G(\cdot, U)$ are sheaves. Composing $U \to \mathrm{Hom}_G(\cdot, U)$ with $\mathcal{F} \to \mathcal{F}({}_GG)$ we get $U \mapsto \mathrm{Hom}_G({}_GG, U)$ which is canonically isomorphic to U (namely, a G-equivariant map of ${}_GG$ into U is uniquely determined by the image of 1 in the exact same way as above). Composing in the reverse direction $\mathcal{F} \mapsto \mathcal{F}({}_GG)$ with $U \mapsto \mathrm{Hom}_G(\cdot, U)$ we have to show that the presheaf $\mathrm{Hom}_G(\cdot, \mathcal{F}({}_GG))$ is naturally isomorphic to $\mathcal{F}$, provided that $\mathcal{F}$ is a sheaf. Suppose U is another G-set. Then $\{{}_GG \xrightarrow{\phi_u} U\}_{u\in U}$ (where $\phi_u(g) = g \bullet u$) is a covering of U. Since $\mathcal{F}$ is a sheaf we have the exact sequence:

$$(*) \qquad \mathcal{F}(U) \longrightarrow \prod_{u\in U} \mathcal{F}({}_GG) \rightrightarrows \prod_{u,v\in U} \mathcal{F}({}_GG \times_U {}_GG)$$

Now we note that the middle term is exactly $\mathrm{Mor}(U, \mathcal{F}({}_GG))$ (maps of sets). Since the sequence is exact, we have that $\mathcal{F}(U)$ is the equalizer of the second two arrows. This means it is exactly isomorphic to the subset of morphisms in $\mathrm{Mor}(U, \mathcal{F}({}_GG))$ that commute with the G-action (FIXME?), i.e., $\mathcal{F}(U) \cong \mathrm{Hom}_G(U, \mathcal{F}({}_GG))$. This isomorphisms is clearly functorial in U so we have an isomorphism of sheaves, as desired. Note that in the special case that U is a left G-module rather than just a set, then this process gives an equivalence between left G-modules and sheaves of abelian groups on T_G.

Subsection 2.2. **More about coverings.** Let $\mathcal{C}$ be a site. A morphism of coverings of $\mathcal{C}$ from $\mathcal{U} = \{U_i \to U\}_{i\in I}$ to $\mathcal{V} = \{V_j \to V\}_{j\in J}$ is given by a morphism $U \to V$, a map of sets $\alpha : I \to J$ and for each $i \in I$ a morphism $U_i \to V_{\alpha(i)}$ such that the diagram

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_{\alpha(i)} \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

is commutative. In the special case that $U = V$ and $U \to V$ is the identity we call $\mathcal{U}$ a refinement of the covering $\mathcal{V}$.

Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$, and let $\mathcal{U}$ be a covering in $\mathcal{C}$ as above. Let us use the notation $\mathcal{F}(\mathcal{U})$ to indicate the equalizer

$$\mathcal{F}(\mathcal{U}) = \{(s_i)_i \in \prod_i \mathcal{F}(U_i) \mid \mathrm{pr}_1^* s_i = \mathrm{pr}_2^* s_j \forall i, j \in I\}.$$

There is a canonical map $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$. It is clear that a morphism of coverings $\mathcal{U} \to \mathcal{V}$ induces commutative diagrams

$$\begin{tikzcd} & U_i \arrow[rr] & & V_{\alpha(i)} \, . \\ U_i \times_U U_j \arrow[ur] \arrow[dr] \arrow[rr] & & V_{\alpha(i)} \times_V V_{\alpha(j)} \arrow[ur] \arrow[dr] & \\ & U_j \arrow[rr] & & V_{\alpha(j)} \end{tikzcd}$$

This in turn produces a map $\mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$, compatible with the map $\mathcal{F}(V) \to \mathcal{F}(U)$.

Lemma 2.2.1. *Any two morphisms $f, g : \mathcal{U} \to \mathcal{V}$ of coverings inducing the same morphism $U \to V$ induce the same map $\mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$.*

Proof. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$. The morphism f consists of a map $U \to V$, a map $\alpha\colon I \to J$ and maps $f_i\colon U_i \to V_{\alpha(i)}$. Likewise, g determines a map $\beta\colon I \to J$ and maps $g_i\colon U_i \to V_{\beta(i)}$. As f and g induce the same map $U \to V$, the diagram

$$\begin{tikzcd} & V_{\alpha(i)} \arrow[dr] & \\ U_i \arrow[ur, "f_i"] \arrow[dr, "g_i"'] & & V \\ & V_{\beta(i)} \arrow[ur] & \end{tikzcd}$$

is commutative for every $i \in I$. Hence f and g factor through the fibre product

$$\begin{tikzcd} & V_{\alpha(i)} \\ U_i \arrow[ur, "f_i"] \arrow[r, "\varphi"] \arrow[dr, "g_i"'] & V_{\alpha(i)} \times_V V_{\beta(i)} \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\ & V_{\beta(i)}. \end{tikzcd}$$

Now let $s = (s_j)_j \in \mathcal{F}(\mathcal{V})$. Then for all $i \in I$:

$$(f^*s)_i = f_i^*(s_{\alpha(i)}) = \varphi^*\mathrm{pr}_1^*(s_{\alpha(i)}) = \varphi^*\mathrm{pr}_2^*(s_{\beta(i)}) = g_i^*(s_{\beta(i)}) = (g^*s)_i,$$

where the middle equality is given by the definition of $\mathcal{F}(\mathcal{V})$. This shows that the maps $\mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$ induced by f and g are equal. □

Suppose that $\mathcal{F}$ is a presheaf of abelian groups on a fixed site T. We would like to canonically associate a sheaf $\mathcal{F}^\#$ to $\mathcal{F}$ such that there exists a functorial morphism $\mathcal{F} \to \mathcal{F}^\#$ and such that for any morphism from $\mathcal{F}$ to an abelian sheaf $\mathcal{G}$ there is a unique factorization $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$. The sheaf $\mathcal{F}^\#$ will be called the sheafification of $\mathcal{F}$.

FIXME: Move the following stuff on limits to another file, and make it more general. Find nice lim symbol anyone?

Definition 2.2.2. A directed set is a set S together with a relation $\geq$ which is transitive and reflexive such that for $a, b \in S$ there exists another element $c \in S$ such that $c \geq a$ and $c \geq b$.

A directed system over a directed set S, is given by a set M_s for every $s \in S$ and a map $M_a \to M_b$ for every pair $b \geq a$ such that all the composition $M_a \to M_b \to M_c$ equals the map $M_a \to M_c$ whenever $c \geq b \geq a$. The limit of the directed system is the set $\lim_{s \in S} M_s = \left(\coprod_{a \in S} M_a\right)/\sim$. Here, if $m \in M_a$ and $m' \in M_{a'}$, then $m \sim m'$ if and only if m and m' map to the same element in some M_b for some b with $b \geq a$

and $b \geq a'$. If the system is in the category of abelian groups then the limit has the structure of an abelian group.

Let $\mathcal{J}_U$ be the set of all coverings of U. It is not hard to check that $\mathcal{J}_U$ with morphisms being morphisms of coverings over U, is a category. It is also clear that given any two coverings of U, $\mathcal{U}_1$ and $\mathcal{U}_2$, there is another covering refining them both. That is, the covering $\{U_{1i} \times_U U_{2j} \to U\}$ is a cover of U and the natural projection maps give the refinements: it is exactly conditions (2) and (3) in 2.1.1 that allow us to conclude that this is a cover. Now, by the above remarks, we see that $\mathcal{J}_U$ is a directed set, where we say that $\{U_i \to U\} \geq \{V_j \to U\}$ if and only if $\{U_i \to U\}$ is a refinement of $\{V_j \to U\}$. Lemma 2.2.1 tells us that $\mathcal{U} \mapsto \mathcal{F}(\mathcal{U})$ is a directed system over $\mathcal{J}(U)$, if we define, for $\mathcal{U} \geq \mathcal{V}$ the map $\mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{V})$ to be induced from any morphism of coverings of U. Hence we can take the direct limit over the set of coverings of U. Thus we define

$$\mathcal{F}^\dagger(U) = \lim_{\mathcal{U}} \mathcal{F}(\mathcal{U})$$

where the limit is over the directed set of coverings $\mathcal{J}_U$. This is sometimes denoted $\check{H}^0(U, F)$, ie, the 0th Čech cohomology group.

Finally, we say that $\mathcal{F}$ is separated if, for all coverings of U, $\{U_i \to U$ the map $(F) \to \prod (F)(U_i)$ is injective.

Theorem 2.2.3. *With $\mathcal{F}$ as above*

(1) *$\mathcal{F}^\dagger$ is separated*
(2) *If $\mathcal{F}$ is separated, then $\mathcal{F}^\dagger$ is a sheaf.*
(3) *$\mathcal{F}^{\dagger\dagger}$ is always a sheaf.*

Proof. FIXME. □

FIXME. Discuss the more general case when $\mathcal{F}$ may not be a sheaf with values in $\mathcal{A}b$.

Section 3. Representable sheaves

FIXME. Talk about representable presheaves, canonical topology and representable sheaves.

Section 4. Morphisms of sites

FIXME. Talk about continuous functors, and explain the condition that leads to the correct functoriality on sheaves (i.e., exactness of the pullback functor). It makes sense to not always assume this holds.

Section 5. Topoi

The topos associated to a site $\mathcal{C}$ is its "category" of sheaves of sets. Conversely, any topos is equivalent to such a "category" of sheaves. Our conventions do not allow us to talk about topoi. Of course we can choose a large cardinal α and consider the category of sheaves of sets $\mathrm{Sh}_\alpha(\mathcal{C})$ contained in α, but this does not have the same flavor.

FIXME. What are topoi? What are morphisms of topoi? Do we need them? (Yes, in a way.)

As a result some of the discussion in this project uses sites in places where it might be more convenient to use the language of topoi. We discuss a few of these "inconveniences" in this section.

Subsection 5.1. **Sites and points.** A point of a topos $\mathcal{S}$ is a morphism of topoi from Sets to $\mathcal{S}$. As discussed above we do not use this definition. In stead, we somewhat akwardly define a point as follows. A point is a functor $p : \mathcal{C} \to \text{Sets}$ such that

(1) if $V \times_U W$ exists then $p(V \times_U W) = p(V) \times_{p(U)} p(W)$,
(2) if $\{U_i \to U\}$ is a covering, then $\coprod_i p(U_i) \to p(U)$ is surjective,
(3) for any $x \in p(U)$ and $y \in p(V)$ there exists a $z \in p(W)$ and morphisms $\alpha : W \to U$, $\beta : W \to V$ such that $p(\alpha)(z) = x$, and $p(\beta)(z) = y$, and
(4) for any pair of morphisms $f, g : V \to U$, and $y \in p(V)$ such that $p(f)(x) = p(g)(x)$, there exists a $h : W \to V$, $z \in p(W)$ such that $p(h)(z) = y$ and $g \circ h = f \circ h$.

Once we have this, then we can define the stalk of a (pre)sheaf $\mathcal{F}$ at p as follows

$$\mathcal{F}_p = \lim_{(U,x)} \mathcal{F},$$

where the limit is over the category of pairs $\{(U, x) \mid U \in \text{Ob}(\mathcal{C}), x \in p(U)$. The conditions above imply this is a FIXME filtered limit. This implies that taking stalks is an exact functor. FIXME: Need a section on limits.

Lemma 5.1.1. *In the situation above we have $p(U) = (U^{++})_p$. FIXME: notation.*

Proof. FIXME. □

We say that a site $\mathcal{C}$ has enough points if the following equivalence is true for every morphism of sheaves of sets $\phi : \mathcal{F} \to \mathcal{G}$:

$$\phi \text{ is injective} \Leftrightarrow \forall p, \phi_p \text{ is injective}$$

This will then imply the same thing for "bijective" and "surjective", and it allows you to check exactness of sequences of sheaves of abelian groups on stalks. (FIXME: explain?) Often sites that we work with have enough points and it is easier to work with them, e.g., it is fairly easy to construct injective sheaves of abelian groups on such a site.

To continue reading,

(1) visit the next section: Flat descent for quasi-coherent sheaves, Section 1, or
(2) go back to the table of contents: index.html#contents.

References

[Art62] Micheal Artin. Grothendieck topologies. page 134 pages, 1962.
[MA71] J.L. Verdier M. Artin, A. Grothendieck. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III*, volume 269, 270, 305 of *Lecture Notes in Mathematics*. Springer, 1971.