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SCHEMES AS STACKS AND REPRESENTABILITY

Contents

Section 1. Introduction

In this document we explain how we will think of schemes as stacks over the category of affine schemes.

Section 2. Affine schemes, schemes, stacks representable by a scheme

You can skip the first two subsections for sure.

Subsection 2.1. Locally ringed spaces. A locally ringed space $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ all of whose stalks are local rings. Morphisms in the category of locally ringed spaces are maps of pairs $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ so that all the induced ring maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are local ring maps.

A reference for this section is [DG67], I.

Subsection 2.2. Affine schemes. An affine scheme is a locally ringed space isomorphic to a locally ringed space of the form $\mathrm{Spec}(A)$, for some commutative (unital) ring A. (Note that A can be the zero ring in which case $\mathrm{Spec}(A)$ is the empty space.) As a set $\mathrm{Spec}(A)$ is the set of prime ideals of A. The topology on $\mathrm{Spec}(A)$ is the unique one that has a basis of opens of the form $D(f) = \{\wp \in \mathrm{Spec}(A) \mid f \notin \wp\}$, $f \in A$. The structure sheaf $\mathcal{O} = \mathcal{O}_{\mathrm{Spec}(A)}$ is the unique sheaf of rings such that (1) $\Gamma(D(f), \mathcal{O}) = A_f$ and (2) the restriction map $\Gamma(D(f), \mathcal{O}) \to \Gamma(D(fg), \mathcal{O})$ is the canonical map $A_f \to A_{fg}$.

A morphism of affine schemes is a morphism in the category of locally ringed spaces.

Subsection 2.3. **The category of affine schemes.** It should be clear what the category of affine schemes is, except for a little bit of set-theoretical discussion. We will use the notation Aff to denote this category. Our approach is to use only categories which are sets. Thus we will choose a supply of affines and work with this. For a precise mathematical discussion, see Subsection 2.3.1.

The topology on Aff will be the fppf topology. A covering is given by a finite family of maps $\{U_i \to U\}$, where each $U_i \to U$ is a finitely presented flat morphism of affines, and $\coprod U_i \to U$ is surjective.

Sometimes we consider Aff with other topologies, such as the etale, Zariski, or fpqc topologies. Notation Aff_{etale}, etc. FIXME. Put in internal reference to topology discussion.

Subsubsection 2.3.1. *Sets of affine schemes.* Choose an ordinal α and denote Aff_α the category of affine schemes which are elements of V_α, see Sets,Subsection 2.1. So there is a theory of algebraic stacks for any α. There are some minimal conditions on α needed to imply that Aff_α is a site. These minimal required properties are expressed in the following lemma.

Lemma 2.3.1. *For any set S may choose an ordinal α with $S \in V_\alpha$ such that Aff_α has (finite) fibre products, and finite disjoint unions. In addition we may assume that for any finitely presented morphism of affines $X \to Y$, such that $Y \in \mathit{Aff}_\alpha$, there exists an affine $X' \in Ob(\mathit{Aff}_\alpha)$ such that $X' \cong X$.*

Proof. Consider the following statement: "For any finite directed graph Γ, for any assignment $v \mapsto F(v)$, $\forall v \in \mathrm{Vertices}(\Gamma)$, where $F(v)$ is an affine scheme, and any assignment $\big(e : v_1 \to v_2\big) \mapsto \big(F(e) : F(v_1) \to F(v_2)\big)$, $\forall e \in \mathrm{Edges}(\Gamma)$ where $F(e)$ is a morphism of affine schemes, there exists an affine scheme X and morphisms $f(v) : X \to F(v)$, $\forall v \in \mathrm{Vertices}(\Gamma)$ such that $f(v_2) = F(e) \circ f(v_1)$, $\forall \big(e : v_1 \to v_2\big) \in \mathrm{Edges}(\Gamma)$, such that $(X, \{f(v)\}_{v \in \mathrm{Vertices}(\Gamma)})$ is universal among all such." This statement says that finite limits exist for affine schemes. It is proved in a standard way (for example by turning it into ring theory).

On the other hand, upon formalizing the statement we obtain a provable formula $\phi(\Gamma, F)$ of ZFC set theory. Hence, according to the reflection principle, see Sets, Lemma 3 there exists an ordinal α such that the formula is true in V_α: If you take $\Gamma \in V_\alpha$ and the $F(v)$ to be in Aff_α, then you can find a solution $(X, \{f(v)\}_{v \in \mathrm{Vertices}(\Gamma)})$ with X in V_α. This takes care of the statement about fibre products. (Of course as soon as α is infinite then every graph is isomorphic to a graph in V_α; we can also simply apriori require this for V_α.).

We can similarly write out the condition of the existence of disjoint unions as a set theory formula, and similarly the existence of the affine X' given $X \to Y$. The reflection principle states we can have S inside of V_α as well. □

Clearly, we may assume that Aff_α is closed under any reasonable operation (see Sets, Section 3). Of course, whenever we require such a condition we will need to write out the proof that this is so.

So, in the following we will work with stacks (or categories) over Aff_α[1]. If $\alpha < \beta$, then there is an inclusion $\mathrm{Aff}_\alpha \subset \mathrm{Aff}_\beta$, and hence any category over Aff_β gives rise

[1]As per our general philosophy, if we ever need an actual 2-category of stacks, we also choose another cardinal γ and consider only those categories over Aff_α contained in V_γ.

to a category over Aff_α. But this is not the correct thing to do when studying algebraic stacks. Instead we want to show that algebraic stacks over Aff_α give rise to algebraic stacks over Aff_β. In other words we will need a theorem saying that the 2-category of algebraic stacks over Aff_α is equivalent to a full sub-2-category of algebraic stacks over Aff_β. Here it is.

FIXME. Improve the theorem below and move it to a more appropriate spot.

Theorem 2.3.2. *Suppose that $p : \mathcal{S} \to \mathit{Aff}_\alpha$ is an algebraic stack. Let $\beta > \alpha$. Then there exists an algebraic stack $p' : \mathcal{S}' \to \mathit{Aff}_\beta$ and an equivalence*

$$\begin{tikzcd} (p')^{-1}(\mathit{Aff}_\alpha) \arrow[rr, "c"] \arrow[dr, "p'"'] & & \mathcal{S} \arrow[dl, "p"] \\ & \mathit{Aff}_\alpha . & \end{tikzcd}$$

The pair $((\mathcal{S}', p'), c)$ is well determined up to a 1-isomorphism (which is itself unique up to unique 2-isomorphism).

Proof. FIXME. Hint. Choose a representation (in $\mathrm{Stacks}/\mathrm{Aff}_\alpha$) $\mathcal{S} = [\mathcal{X}/\mathcal{R}]$, with $\mathcal{X}$ representable by a scheme X and $\mathcal{R}$ representable by an algebraic space. Choose a presentation $\mathcal{R} = [\mathcal{U}/\mathcal{R}_\mathcal{U}]$ where $\mathcal{U}$ and $\mathcal{R}_\mathcal{U}$ are representable by schemes U and R_U. Now define (in $\mathrm{Stacks}/\mathrm{Aff}_\beta$) $\mathcal{U}'$ to be the stack associated to U, $\mathcal{R}'_\mathcal{U}$ to be the stack associated to R_U, $\mathcal{R}'$ the stack $\mathcal{R}' = [\mathcal{U}'/\mathcal{R}'_\mathcal{U}]$, $\mathcal{X}'$ the stack associated to X, and finally $\mathcal{S}' = [\mathcal{X}'/\mathcal{R}']$. ☐

From now on Aff will denote a category of affines Aff_α such as in Lemma 2.3.1. By the theorem above we may increase α whenever this is needed.

Remark 2.3.3. There is another approach. Allow yourself to enlarge α at any moment. Think of every statement in the text as being preceded by "There exist arbitrarily large α such that".

Subsection 2.4. **Schemes.** We recall the definition of a scheme.

A scheme $(X, \mathcal{O}_X)$ is a locally ringed space with the property that every point has a neighbourhood which is an affine scheme.

A scheme X gives rise to a functor (or presheaf)

$$\mathrm{Aff}^{\mathrm{opp}} \xrightarrow{h_X} \mathrm{Sets}, \qquad U \longmapsto \mathrm{Mor}(U, X).$$

The usual Yoneda lemma tells us that we can recover the scheme from this functor.

Lemma 2.4.1. *Suppose that X, Y are schemes with that have open coverings by affines isomorphic to objects of Aff. Then $\mathit{Mor}(X, Y) = \mathit{Mor}(h_X, h_Y)$.*

Proof. FIXME. ☐

Subsection 2.5. **Stacks representable by a scheme.** In Categories, Definition 3.2.4 we defined the notion of a representable category fibred in groupoids. This, applied to a stack (or a category) over Aff will define the notion of a stack representable by an affine scheme.

Here is the formal definition of a category over Aff representable by a scheme. Please also see the informal discussion below.

Definition 2.5.1. A category fibred in groupoids $p : \mathcal{S} \to \mathrm{Aff}$ is called representable by a scheme, if the following conditions are satisfied:

(1) all fibre categories $\mathcal{S}_U$ are setlike, and
(2) the presheaf $U \mapsto \mathrm{Ob}(\mathcal{S}_U)/\cong$ is is isomorphic to h_S for a scheme S as in Lemma 2.4.1.

Lemma 2.5.2. *If $\mathcal{S} \to \mathit{Aff}$ is representable by a scheme then $\mathcal{S}$ is a stack over Aff.*

Proof. FIXME. □

Example 2.5.3. Let X be a scheme that has a covering by open affines which are isomorphic to objects of Aff. There is a standard stack over Aff representable by X, namely the stack of affines over X. Compare Categories, Example 2.1.6. This stack will be denoted Aff/X, and it is described as follows.

(1) An object of Aff/X is a morphism of schemes $U \to X$, with $U \in \mathrm{Ob}(\mathrm{Aff})$.
(2) A morphism between $U \to X$ and $V \to X$ is a commutative diagram

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X. & \end{tikzcd}$$

(3) The functor $(\mathrm{Aff}/X) \to \mathrm{Aff}$ maps $U \to X$ to U.

It is clear from the definition that Aff/X is representable by a scheme.

The construction is clearly functorial in X, so that a morphism of schemes $f : X \to Y$ induces a morphisms of stacks $\mathrm{Aff}/X \to \mathrm{Aff}/Y$. FIXME: more?

Situation 2.5.4. The following situation will appear repeatedly in the text. Suppose that $\mathcal{S} \to \mathrm{Aff}$ is a stack representable by a scheme. If we say the scheme S represents $\mathcal{S}$, then we mean that besides being given the scheme S, we are given an equivalence $j : \mathcal{S} \to \mathrm{Aff}/S$ of stacks over Aff.

Lemma 2.5.5. *Suppose that the stacks $\mathcal{X}$, $\mathcal{Y}$ are represented by the schemes X and Y. For any morphism of stacks $F : \mathcal{X} \to \mathcal{Y}$ there is a unique morphism of schemes $f : X \to Y$ such that the diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] \arrow[d, "j"'] & \mathcal{Y} \arrow[d, "j"] \\ \mathit{Aff}/X \arrow[r, "f"] & \mathit{Aff}/Y \end{tikzcd}$$

2-commutes and then the diagram actually commutes.

Proof. FIXME. □

Section 3. Morphisms representable by schemes

In this section we define the notion of moprhisms of stacks over Aff representable by schemes.

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Subsection 3.1. **Definition.** Here is the formal definition. Please also see the informal discussion below.

Definition 3.1.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over Aff. We say f is representable by schemes if for every stack $\mathcal{S}$ representable by a scheme (see Definition 2.5.1), and every morphism $\mathcal{U} \to \mathcal{Y}$, the 2-fibre product $\mathcal{S} \times_{\mathcal{Y}} \mathcal{X}$ is representable by a scheme.

Informal discussion. In the situation of the definition we sometimes say that $\mathcal{X}$ is relatively representable over $\mathcal{Y}$. Suppose that, with the notation of the definition, S represents $\mathcal{S}$ and W represents $\mathcal{S} \times_{\mathcal{Y}} \mathcal{X}$. According to Lemma 2.5.5 we get a morphism of schemes $g : W \to S$ and a 2-commutative diagram of stacks

$$\begin{tikzcd}
\mathrm{Aff}/W \arrow[d, "g"] & \mathcal{S} \times_{\mathcal{X}} \mathcal{Y} \arrow[l, "j"'] \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\mathrm{Aff}/S & \mathcal{S} \arrow[l, "j"'] \arrow[r] & \mathcal{X}
\end{tikzcd}$$

FIXME: more.

FIXME. It seems to me that you can define the notion even if $\mathcal{X}$ and $\mathcal{Y}$ are just categories over Aff. Does it make sense in this generality?

Definition 3.1.2. Let P be a property of morphisms of schemes such that if the morphism $f : X \to Y$ has property P, then so does every base change of f. (FIXME: introduce base change.) We say that a morphism of stacks $\mathcal{X} \to \mathcal{Y}$ representable by schemes has property P if for every diagram as above the morphism of schemes $g : W \to S$ has property P.

FIXME. Explain rationale behind this definition: what else could it be?

To continue reading,

(1) visit the next section: Algebraic stacks, Section 1, or
(2) go back to the table of contents: `index.html#contents`.

References

[DG67] Jean Dieudonné and Alexandre Grothendieck. Éléments de géométrie algébrique. *Inst. Hautes Études Sci. Publ. Math.*, 4, 8, 11, 17, 20, 24, 28, 32, 1961-1967.