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INJECTIVES

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Section 1. Introduction

We will use the existence of sufficiently many injectives to do cohomology of abelian sheaves on a site. So we briefly explain why there are enough injectives. At the end we explain the more general story.

Section 2. Existence of injectives in special cases

Grothendieck proved the existence of injectives in great generality in the paper [Gro57]. We will prove this is true for abelian (pre)sheaves on a site.

Subsection 2.1. **Modules.** As an example theorem let us try to prove that there are enough injective modules over a ring R. We start with the fact that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. This we leave to the reader.

For any ring R and any R-module M over R we denote $M^\wedge = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$ with its natural R-module structure. Also we denote $F(M)$ the free module with basis given by the elements of M and we let $F(M) \to M$ be the natural surjection of R-modules.

Note that there is a canonical map $M \to (M^\wedge)^\wedge$. This is injective; you can check this using that $\mathbf{Q}/\mathbf{Z}$ is injective. There is a canonical injection $(M^\wedge)^\wedge \to (F(M^\wedge))^\wedge$. Set $J(M) = (F(M^\wedge))^\wedge$. The composition of the two maps above gives $M \to J(M)$. This will be the desired injection of M into an injective R-module.

Note that $J(M) \cong \prod_{m \in M} R^\wedge$ as an R-module. As the product of injective modules is injective, it suffices to show that $R^\wedge$ is injective. For this you use that $\mathrm{Hom}_R(N, R^\wedge) = \mathrm{Hom}_R(R, N^\wedge)$ and the fact that $(-)^\wedge$ is an exact functor.

The proof above gives us the best possible situation. Not only does every module inject into an injective module, but the construction is completely functorial. Namely, for any map of R-modules $M \to N$ there is an associated morphism $J(M) \to J(N)$

making the following diagram commute:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ J(M) \arrow[r] & J(N) \end{tikzcd}$$

This the kind of construction we would like to have in general.

Subsubsection 2.1.1. *Categories of modules.* As a consequence we obtain a category of modules with a canonical resolution. For an ordinal α we denote $\text{Mod}_{R,\alpha}$ the category of modules contained in V_α (see Sets, Subsection 2.1).

Lemma 2.1.1. *For any given set of R-modules $\{M_i\}_{i\in I}$ there exists an ordinal α such that $M_i \in Ob(Mod_{R,\alpha})$, $\forall i \in I$ and such that for any $M \in Ob(Mod_{R,\alpha})$ we have $J(M) \in Ob(Mod_{R,\alpha})$.*

Proof. Consider the formula $\phi(M)$: "M is an R-module and there exists an R-module N such that $N = J(M)$". Apply the reflection principle to $\phi(M)$, see Theorem 3.1.1. (Use $T = \{M_i\}$.) The result follows. ☐

Some remarks are in order. First we observe that the modules $J(M)$ are injective in the absolute sense, and not only injective in the category $\text{Mod}_{R,\alpha}$. Second, in exactly the same way we can make sure that $\text{Mod}_{R,\alpha}$ has all finite limits, finite direct sums, or countable sums and products, etc. Of course the category $\text{mod}_{R,\alpha}$ never has arbitrary direct sums, which is why working with $\text{mod}_{R,\alpha}$ is somewhat cumbersome.

Subsubsection 2.1.2. *Projective resolutions.* FIXME: Remove probably?
For any set S we let $F(S)$ denote the free R-module on S. Then any left R-module has the following two step resolution

$$F(M \times M) \oplus F(R \times M) \to F(M) \to M \to 0.$$

The first map is given by the rule

$$[m1, m2] \oplus [r, m] \mapsto [m1 + m2] - [m1] - [m2] + [rm] - r[m].$$

The nice thing about this is that it is absolutely canonical. Sometimes we write $S_1 = M \times M \coprod R \times M$ and $S_0 = M$, so that the resolution is $F(S_1) \to F(S_0) \to M \to 0$.

Subsection 2.2. **Abelian presheaves.** Let $\mathcal{C}$ be a category. On the one hand, consider abelian presheaves on $\mathcal{C}$. On the other hand, consider families of abelian groups indexed by elements of $\text{Ob}(\mathcal{C})$; in other words presheaves on the discrete category with underlying set of objects $\text{Ob}(\mathcal{C})$. We will denote presheaves on $\mathcal{C}$ by B and presheaves on $\text{Ob}(\mathcal{C})$ by A. Consider the forgetful functor v, denoted $B \mapsto vB$.

There is a functor u that assigns a presheaf on $\mathcal{C}$ to a presheaf on $\text{Ob}(\mathcal{C})$. It is defined as follows:

$$\Gamma(U, uA) = \prod\nolimits_{U' \to U} A(U').$$

So an element is a family $(a_\phi)_\phi$ with ϕ ranging through all morphisms in $\mathcal{C}$ with target U. The restriction map on uA corresponding to $g : V \to U$ maps our element $(a_\phi)_\phi$ to the element $(a_{g\circ\psi})_\psi$.

There is a canonical surjective map $vuA \to A$ and a canonical map injective map $B \to uvB$. We leave it to the reader to show that

$$\mathrm{Mor}_{\mathrm{PAb}(\mathrm{Ob}(\mathcal{C}))}(B, uA) = \mathrm{Mor}_{\mathrm{PAb}(\mathcal{C})}(vB, A).$$

(Obvious notation.) Thus the pair (u, v) is an example of a pair of adjoint functors. FIXME: Discuss this somewhere. It is clear that u and v are exact functors. It is clear that any presheaf on $\mathrm{Ob}(\mathcal{C})$ has an injective hull. In fact there is a functor J such that $A \mapsto \big(A \to J(A)\big)$ is functorial as in Subsection 2.1. (Namely, $J(A)$ is the assignment $U \mapsto J(A(U))$, where $J(A(U))$ is the functor constructed in Subsection 2.1 for the ring $\mathbf{Z}$ applied to the $\mathbf{Z}$-module $A(U)$.)

Putting all of this together gives us the following procedure for embedding objects B of $\mathrm{PAb}(\mathcal{C})$) into an injective object: $B \to uJ(vB)$.

Proposition 2.2.1. *For abelian presheaves on a category there is a functorial injective hull.*

Subsubsection 2.2.1. *Categories of presheaves of abelian groups.* Arguing as in the proof of Lemma 2.1.1 we obtain a category with an injective resolution functor. For any ordinal α, we use the notation $\mathrm{PAb}(\mathcal{C})_\alpha$ to denote the category of presheaves $\mathcal{F}$ of abelian groups with $\mathcal{F} \in V_\alpha$. See Sets, Subsection 2.1.

Lemma 2.2.2. *Given any set of abelian presheaves $\mathcal{F}_i$, $i \in I$, there exists an ordinal α such that $PAb(\mathcal{C})_\alpha$ contains all of the $\mathcal{F}_i$, and such that there is a functor $PAb(\mathcal{C})_\alpha \to Arrows(PAb(\mathcal{C})_\alpha)$ of the form $\mathcal{F} \mapsto (\mathcal{F} \to J(\mathcal{F}))$ with the property that $\mathcal{F} \to J(\mathcal{F})$ is an injective hull for all $\mathcal{F} \in PAb(\mathcal{C})_\alpha$.*

Proof. FIXME. Very similar to the corresponding lemma for modules. □

Subsection 2.3. **Abelian Sheaves.** Let $\mathcal{C}$ be a site. In this section we prove that there are enough injectives for abelian sheaves on $\mathcal{C}$.

Denote i the forgetfull functor from sheaves to presheaves. Let # denote the sheafification functor, see FIXME. In this subsection we will use that $i(\mathcal{F})^\# = \mathcal{F}$, see FIXME. Finally, let $\mathcal{F} \to J(\mathcal{F})$ denote the canonical embedding into an injective presheaf we found in Subsection 2.2.

For any sheaf $\mathcal{F}$ in $\mathrm{Ab}(\mathcal{C})$ and any ordinal β we define a sheaf $J_\beta(\mathcal{F})$ by transfinite induction. FIXME: explain transfinite induction in src/sets.tex. First we set $J_0(\mathcal{F}) = \mathcal{F}$. We define $J_1(\mathcal{F}) = J(i(\mathcal{F}))^\#$; there is a map $\mathcal{F} = i(\mathcal{F})^\# \to J(i\mathcal{F})^\#$ by functoriality of #. This map $\mathcal{F} \to J_1(\mathcal{F})$ is injective as # is exact. We set $J_{\alpha+1} = J_1(J_\alpha)$, and for a limit ordinal β, we define

$$J_\beta(B) = \lim_{\alpha<\beta} J_\alpha(B).$$

FIXME: limit notation.

Lemma 2.3.1. *With notation as above. Suppose that $\mathcal{G}_1 \to \mathcal{G}_2$ is an injective map of abelian sheaves on $\mathcal{C}$. Let α be an ordinal and let $\mathcal{G}_1 \to J_\alpha(\mathcal{F})$ be a morphism of sheaves. There exists a morphism $\mathcal{G}_2 \to J_{\alpha+1}(\mathcal{F})$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r] \arrow[d] & \mathcal{G}_2 \arrow[d] \\ J_\alpha(\mathcal{F}) \arrow[r] & J_{\alpha+1}(\mathcal{F}) \end{tikzcd}$$

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Proof. FIXME. □

This lemma says that somehow the system $\{J_\alpha(\mathcal{F})\}$ is the injective hull of $\mathcal{F}$ that we are looking for. Of course we cannot take the limit over all α because they form a class and not a set. However, the idea is now that you don't have to check injectivity on all injections $\mathcal{G}_1 \to \mathcal{G}_2$, plus the following lemma.

Lemma 2.3.2. *Suppose that $\mathcal{G}_i$, $i \in I$ is set of sheaves of abelian groups on $\mathcal{C}$. There exists an ordinal β such that for any sheaf $\mathcal{F}$, any $i \in I$, and any map $\varphi : \mathcal{G}_i \to J_\beta(\mathcal{F})$ there exists an $\alpha < \beta$ such that φ factors through $J_\alpha(\mathcal{F})$.*

Proof. (You can reduce this to the case of a single sheaf $\mathcal{G}$ by taking the direct sum of all the $\mathcal{G}_i$.) FIXME. Hint: First suppose that $T = \lim_{\alpha<\beta} T_\alpha$ is a limit of sets and that $\varphi : S \to T$ is a map of sets. Then φ lifts to a map into T_α for some $\alpha < \beta$ provided that β is not a limit of ordinals indexed by S. In other words, you pick β to be a cardinal with cofinality $cf(\beta)$ bigger than the cardinality of S; for example you can take $\beta = \aleph_{\alpha+1}$. Reference? Use this and some argument for equalizers to get through. □

Recall that for an object X of $\mathcal{C}$ we denote $\mathbf{Z}_X$ the presheaf of abelian groups $\Gamma(U, \mathbf{Z}_X) = \oplus_{U\to X}\mathbf{Z}$. FIXME: should be introduced in src/sites.tex. The sheaf associated to this presheaf is denoted $\mathbf{Z}_X^\#$.

Lemma 2.3.3. *Suppose $\mathcal{J}$ is a sheaf of abelian groups with the following property: For all $X \in Ob(\mathcal{C})$, for all subsheaves $\mathcal{G} \subset \mathbf{Z}_X^\#$ and for all morphisms $\varphi : \mathcal{G} \to \mathcal{J}$, there exists an morphism $\mathbf{Z}_X^\# \to \mathcal{J}$ extending φ. Then $\mathcal{J}$ is an injective sheaf of abelian groups.*

Proof. FIXME. □

Theorem 2.3.4. *The category of abelian sheaves has enough injectives (in the strongest possible sense).*

Proof. FIXME. Idea: Let $\mathcal{G}_i$, $i \in I$ be a set of abelian sheaves such that every subsheaf of every $\mathbf{Z}_X^\#$ occurs as one of the $\mathcal{G}_i$. Apply Lemma 2.3.2 to this collection to get an ordinal β. We claim that for any sheaf of abelian groups $\mathcal{F}$ the map $\mathcal{F} \to J_\beta(\mathcal{F})$ is an injection of $\mathcal{F}$ into an injective. Note that by construction the assigment $\mathcal{F} \mapsto \big(\mathcal{F} \to J_\beta(\mathcal{F})\big)$ is functorial.
The proof of the claim comes from the fact that by Lemma 2.3.3 it suffices to extend any morphism $\gamma : \mathcal{G} \to J_\beta(\mathcal{F})$ from a subsheaf $\mathcal{G}$ of some $\mathbf{Z}_X^\#$ to all of $\mathbf{Z}_X^\#$. Then by Lemma 2.3.2 the map γ lifts into $J_\alpha(\mathcal{F})$ for some $\alpha < \beta$. Finally, we apply Lemma 2.3.1 to get the desired extension of γ to a morphism into $J_{\alpha+1}(\mathcal{F}) \to J_\beta(\mathcal{F})$. □

Subsubsection 2.3.1. *Categories of abelian sheaves.* Again we obtain a result concerning the existence of a category preserved by the functorial assigment $\mathcal{F} \mapsto \big(\mathcal{F} \to J_\beta(\mathcal{F})\big)$ described in Theorem 2.3.4. As is usual, for an ordinal α, we denote $\mathrm{Ab}(\mathcal{C})_\alpha$ the category of abelian sheaves on $\mathcal{C}$ which are elements of V_α.

Lemma 2.3.5. *Given any set of abelian sheaves $\mathcal{F}_i$, $i \in I$, there exists an ordinal α such that $Ab(\mathcal{C})_\alpha$ contains all of the $\mathcal{F}_i$, and such that there is a functor $Ab(\mathcal{C})_\alpha \to Arrows(Ab(\mathcal{C})_\alpha)$ of the form $\mathcal{F} \mapsto (\mathcal{F} \to J(\mathcal{F}))$ with the property that $\mathcal{F} \to J(\mathcal{F})$ is an injective hull for all $\mathcal{F} \in Ab(\mathcal{C})_\alpha$.*

Proof. FIXME. Very similar to the corresponding lemma for modules. □

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Section 3. Grothendieck categories and injectives

Here we can talk in general about generators, limits and all that good stuff. This will possibly be needed later on anyway. FIXME.

To continue reading,

(1) visit the next section: Hypercoverings, Section 1, or
(2) go back to the table of contents: `index.html#contents`.

References

[Gro57] Alexandre Grothendieck. Sur quelques points d'algébre homologique. *Tohoku Mathematical Journal*, 9:119–221, 1957.