Source file: `src/hypercovering.tex`

HYPERCOVERINGS

Contents

Section 1. Introduction

Hypercoverings can be used to compute cohomology of abelian sheaves on sites without recourse to injective resolutions. See [MA71, Expose V, Sec. 7]. A nice manuscript on cohomological descent is the text by Brian Conrad, see `http://www.math.lsa.umich.edu/~bdconrad/papers/hypercover.pdf`. Probably it is useless to try to improve on Brian's article, so we look at the question a little differently (more naively).

Section 2. Definitions

Let $\mathcal{C}$ be a category. Let Δ be the category of finite ordered sets with objects $[0] = \{0\}, [1] = \{0, 1\}, [2] = \{0, 1, 2\}, \ldots$ and order preserving maps. A simplicial object $U_\bullet$ of $\mathcal{C}$ is a contravariant functor $U_\bullet : \Delta \to \mathcal{C}$. This means there are objects $U_0, U_1, U_2, \ldots$ and morphisms $U_\bullet(\varphi) : U_n \to U_m$, where φ is any order preserving map $\varphi : [m] \to [n]$.

In particular there is a unique morphism $U_0 \to U_n$ and there are exactly $n+1$ morphisms $U_n \to U_0$ corresponding to the $n+1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects.

Definition 2.0.1. For any integer $n \geq 1$, and any $0 \leq j \leq n$ we let $d^n_j : [n-1] \to [n]$ denote the injective order preserving map skipping j. For any integer $n \geq 0$, and any $0 \leq j \leq n$ we denote $s^n_j : [n+1] \to [n]$ the surjective order preserving map with $(s^n_j)^{-1}(\{j\}) = \{j, j+1\}$.

We get a unique morphism $U_\bullet(s^0_0) : U_0 \to U_1$ and two morphisms $U_\bullet(d^1_0), U_\bullet(d^1_1) : U_1 \to U_0$. There are two morphisms $U_\bullet(s^1_0), U_\bullet(s^1_1) : U_1 \to U_2$ and three morphisms $U_\bullet(d^2_0), U_\bullet(d^2_1), U_\bullet(d^2_2) : U_3 \to U_2$. And so on. FIXME: This notation...

FIXME: Much more.

Example 2.0.2. (1) The simplest example is the *constant* simplicial object with value $X \in \mathrm{Ob}(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

(2) Suppose that $Y \to X$ is a morphism of C such that all the fibred products

$Y^n_{/X}Y = \times_X Y \times_X \ldots Y$ exist. Then we set $U_n = Y^{n+1}_{/X}$, and we let $s : [n] \to [m]$ correspond to the map (on "coordinates") $(y_0, \ldots, y_m) \mapsto (y_{s(0)}, \ldots, y_{s(n)})$.

Subsection 2.1. **Goals.** Assume that $\mathcal{C}$ is a site with the property that the set of coverings consisting of 1 morphism is cofinal. Let $\mathcal{F}$ be a sheaf of abelian groups on the site $\mathcal{C}$ which is assumed to have the property that the set of coverings consisting of 1 morphism is cofinal. Choose an injective resolution $\mathcal{F} \to \mathcal{J}^\bullet$ (for example a canonical one, see Injectives, Subsection 2.3). Let X be an object of $\mathcal{C}$. We want to compute $R\Gamma(X, \mathcal{F}) = \Gamma(X, \mathcal{J}^\bullet)$ or at least the cohomology groups $H^j(X, \mathcal{F})$. The idea is to construct simplicial objects $U_\bullet$ augmented towards X, so $U_\bullet \to X$, such that

$$R\Gamma(X, \mathcal{F}) = \mathrm{Tot}(R\Gamma(U_\bullet, \mathcal{J}^\bullet)) \tag{$*$}$$

is a quasi-isomorphism (for any $\mathcal{F}$). On the right hand side this is the total complex associated to the double complex. (The maps are always canonical since we have the resolution over all of $\mathcal{C}$.) The complex $\Gamma(U_\bullet, \mathcal{F})$ maps into the complex on the right. We will show that for any element $\eta \in H^j(X, \mathcal{F})$ there exists a choice of $U_\bullet \to X$ such that η comes from an element in $H^j(U_\bullet, \mathcal{F})$. This is a first step and it already allows us to define cup products for example. The starting point is the following.

Lemma 2.1.1. *Suppose that $\{Y \to X\}$ is a covering in the topology of $\mathcal{C}$. Let $U_n = Y^n_{/X}$ be the simplicial object defined in Example 2.0.2. The augmentation $U_\bullet \to X$ has the property that $(*)$ is a quasi-isomorphism for all $\mathcal{F}$.*

Proof. FIXME. □

Subsection 2.2. **Making simplicial objects.** Suppose that $U_\bullet$ is a simplicial object of $\mathcal{C}$. Now let $n \geq 0$ and let $V \to U_n$ be a representable morphism of $\mathcal{C}$. This means that the fibre products $V \times_{U_n} W$ exist for all morphisms $W \to U_n$.

For any m consider the fibre product (over U_m)

$$U'_m = \prod\nolimits_{\varphi \in \mathrm{Mor}_\Delta([n],[m])} V \times_{U_n, U_\bullet(\varphi)} U_m.$$

By our assumption on the morphism $V \to U_n$ this fibre product exists. For any $\psi : [m1] \to [m2]$ there is a canonical morphism $U'_{m2} \to U'_{m1}$ coming from the map $\mathrm{Mor}_\Delta([n],[m1]) \to \mathrm{Mor}_\Delta([n],[m2]), \varphi \mapsto \varphi \circ \psi$, the identity map on V and the canonical map $U_\bullet(\psi) : U_{m2} \to U_{m1}$

Clearly, these data give rise to a simplicial object $U'_\bullet$ in $\mathcal{C}$. The natural morphisms $U'_m \to U_m$ give rise to a morphism of simplicial objects $U'_\bullet \to U_\bullet$. Note that the morphism $U'_n \to U_n$ factors throught the morphism $V \to U_n$ by projection onto the factor corresponding to $\varphi = \mathrm{id}_{[n]}$. Also, note that if $\mathcal{C}$ is a site and if $\{V \to U_n\}$ is a covering in the site then for any m it is true that $\{U'_m \to U_m\}$ is a covering. This proves the following lemma.

Lemma 2.2.1. *Suppose that $U_\bullet$ and $V \to U_n$ are as above such that $\{V \to U_n\}$ is a covering for the topology on the site $\mathcal{C}$. The morphism of simplicial objects $U'_\bullet \to U_\bullet$ constructed above has the following properties: (1) The morphism $U'_n \to U_n$ factors trough $V \to U_n$. (2) For any m the set $\{U'_m \to U_m\}$ is a covering in the topology of $\mathcal{C}$.*

Source file: `src/hypercovering.tex`

Subsection 2.3. **Doubly simplicial stuff.** A doubly simplicial object of $\mathcal{C}$ is a functor $U_{\bullet,\bullet} : (\Delta \times \Delta)^\circ \to \mathcal{C}$. By subdividing we can make this into a simplicial object $W(U_{\bullet,\bullet})$ with the same cohomology. FIXME: Explain this.
Suppose that $U'_\bullet \to U_\bullet$ is a morphism of simplicial objects of $\mathcal{C}$ such that each of the morphisms $U'_n \to U_n$ is representable. Then we can construct a doubly-simplicial object $U'_{\bullet,\bullet}$ by setting $U'_{n,0} = U'_n$,

$$U'_{n,1} = U'_n \times_{U_n} U'_n,$$

etc. Compare Example 2.0.2. Out of this object we can construct a single simplicial object $W(U'_{\bullet,\bullet})$ as explained above. Construct the natural morphism of simplicial objects $W(U'_{\bullet,\bullet}) \to U_\bullet$.

Lemma 2.3.1. *Suppose that every $\{U'_n \to U_n\}$ is a covering for the topology of $\mathcal{C}$. Suppose that $\mathcal{F}$ is a sheaf on $\mathcal{C}$. Then there is a natural morphism of complexes*

$$R\Gamma(U_\bullet, \mathcal{F}) \to R\Gamma(W(U_{\bullet,\bullet}), \mathcal{F})$$

which is a quasi-isomorphism. FIXME: Something like this in any case.

Section 3. The general case

Mention how things work more generally, for example if $\mathcal{C}$ does not have the property that coverings consisting of a single map are cofinal. State the theorem in the correct generality.

To continue reading,

(1) visit the next section: Stacks, Section 1, or
(2) go back to the table of contents: `index.html#contents`.

References

[MA71] J.L. Verdier M. Artin, A. Grothendieck. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III*, volume 269, 270, 305 of *Lecture Notes in Mathematics*. Springer, 1971.