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THE ÉTALE TOPOLOGY ON SCHEMES

Abstract. In this Chapter, we study étale morphisms of schemes. Our principal goal is to equip the reader with enough (commutative) algebraic tools to approach a treatise on étale cohomology. An auxiliary goal is to provide enough evidence to ensure that the reader stops calling the phrase "the étale topology of schemes" an exercise in general nonsense, if (s)he does indulge in such blasphemy.

Contents

Section 1. Introduction

Almost all the material presented here is taken, without too many modifications, from [Gro71] and [BLR90]. Assuming certain standard results in algebraic geometry (and therefore commutative algebra), we have tried to provide detailed proofs of most of the claims we make. However, as is the bane of the subject, it's almost impossible to provide fully detailed proofs (say, as seen in early undergraduate courses) while maintaining brevity. It is nevertheless hoped that the proofs provided here give more than enough to the reader to reconstruct the entire proof.

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Section 2. Notation and conventions

All rings will be commutative with 1 and, more restrictively, Noetherian. Therefore all schemes will be assumed to be locally Noetherian. If A is a local ring, we will denote its maximal ideal by $r(A)$ and its residue class field by $k(A)$. A morphism of local rings $f : A \to B$ is a ring homomorphism such that $f(r(A)) \subset r(B)$. The completion of a local ring A with the $r(A)$-adic topology is denoted by $\widehat{A}$.

FIXME: Remove Noetherian hypotheses.

Section 3. Unramified morphisms

Subsection 3.1. **Definition and sorites.** We first define the notion of unramified morphisms for local rings, and then globalise it to get one for arbitrary schemes. Along the way, we mention a few sorites which can be easily verified.

Definition 3.1.1. A morphism $f : A \to B$ of local rings is said to be unramified if $f(r(A))B = r(B)$ and $k(B)$ is a finite separable extension of $k(A)$.

It is clear that a morphism $f : A \to B$ of local rings is unramified if and only if $\widehat{f} : \widehat{A} \to \widehat{B}$ is unramified. By basic properties of complete local rings, this also implies that $\widehat{B}$ is a finite $\widehat{A}$ module. Moreover, if $k(A)$ is separably closed, it is easy to see that $\widehat{A} \to \widehat{B}$ is actually surjective. More generally, if $k(B)$ is the trivial extension of $k(A)$, $\widehat{B}$ is a quotient of $\widehat{A}$. Lastly, if A and B are complete discrete valuation rings, $f : A \to B$ is unramified if and only the uniformizer for A is also a uniformizer for B. Thus, this definition agrees with the definition in number theory.

Definition 3.1.2. A morphism $f : X \to Y$ of schemes is said to be unramified at $x \in X$ if it is of finite type at x and the associated morphism of local rings at x ($\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$) is unramified. The morphism $f : X \to Y$ is said to be unramified if it is unramified at all points of x (and therefore is locally of finite type).

By definition, it follows that unramifiedness is local on the source and the target. It is easy to verify that unramified morphisms are stable under base change and composition. One can easily see that quasi-compact unramified morphisms of schemes are quasi-finite (and therefore have relative dimension 0). An important, but once again easily verified, observation is that a morphism that is locally of finite type is unramified if and only if all its fibres are unramified. That is, unramifiedness can be checked on the fibres of a morphism locally of finite type.

Subsection 3.2. **Three other equivalent definitions.**

Theorem 3.2.1. *Let $f : X \to Y$ be a morphism locally of finite type. Let x be a point of X. The following are equivalent*

(1) *f is unramified at x,*
(2) *$\Omega^1_{X/Y}$ is trivial at x,*
(3) *There exists open neighbourhoods U of x and V of $f(x)$, and a V-morphism $U \to \mathbf{A}^n_V$ which is closed immersion defined by a quasi-coherent sheaf of ideals $\mathcal{J}$ such that the differentials $\{dg | g \in \Gamma(\mathbf{A}^n_V, \mathcal{J})\}$ span $\Omega^1_{\mathbf{A}^n_V/V}$ at x,*
(4) *The diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is a local isomorphism at x.*

Proof. $1 \iff 2$: For the forward implication, after taking sufficiently small open sets about x and $f(x)$, we may assume that X and Y are affine (the formation of the module of Kähler differentials is compatible with base change and taking open subsets of the source). Note that this automatically forces f to be of finite type. By Nakayama's lemma, it suffices to show that the fibre of $\Omega^1_{X/Y}$ at x is trivial. Thus, by replacing $X \to Y$ with its fibre over $f(x)$, we reduce to the case that Y is a field. Now, if $X = \mathrm{Spec}(A)$, A is a finite separable k-algebra (k being trivially complete forces A to be finite, and the unramifiedness hypothesis on f forces separability). But now $|X|$ is just a finite union of points with the discrete topology. Thus, we may assume that X itself is the spectrum of a finite separable extension field of k. In this case, it is a well-known result that a finite extension field of a field is separable if and only if the associated vector space of Kähler differentials is zero (check [Mat70], section 27, for the proof). The claim follows.

For the reverse implication, since unramifiedness is a property of the fibres that is local on the source, we once again reduce to the case that Y is a field, and X is the spectrum of a finitely generated k-algebra A. By replacing X with an irreducible component passing through x, we may assume that X is integral (we can do this because if $\Omega^1_{X/Y} = 0$, then $\Omega^1_{Z/Y} = 0$ for any closed immersion $Z \to X$). Thus, X has a function field K. A basic result in commutative algebra says that the rank of $\Omega^1_{X/Y}$ at the generic point (which is also the rank of $\Omega^1_{K/k}$) is at least the transcendence degree of K/k. It follows from the hypothesis that K/k is a finite algebraic extension and, therefore, that $X = \mathrm{Spec}(K)$. We can once again apply the afore-mentioned lemma to conclude that K/k is separable thereby establishing the claim.

$2 \iff 3$: For the forward implication, note that f being locally of finite type gives us (heavily non-canonical) open neighbourhoods U of x and V of $f(x)$ with $f(U) \subset V$, and a closed immersion (over V) $j : U \to \mathbf{A}^n_V$. If j is defined by the sheaf of ideals $\mathcal{J}$, commutative algebra gives an exact sequence

$$j^*(\mathcal{J}/\mathcal{J}^2) \to j^*\Omega^1_{\mathbf{A}^n_V/V} \to \Omega^1_{U/V} \to 0$$

The hypothesis gives us that $\Omega^1_{U/V}$ is trivial at x because the stalk of this sheaf at x is also the stalk of $\Omega^1_{X/Y}$ at x by virtue of the compatibility of the formation of the module with Kähler differentials with restricting to open subsets on both the target and the source. By Nakayama's lemma, we obtained the required implication. The reverse implication follows trivially from the above exact sequence.

$2 \iff 4$: Since both the properties are local on the source and the target, we may assume that X and Y are affine and, consequently, that f is of finite type. The desired implications then follow from the fact that $\Omega^1_{X/Y}$ can be defined as pullback $\Delta^*(\mathcal{J}/\mathcal{J}^2)$ where $\mathcal{J}$ is the sheaf of ideals defining the closed immersion $X \to X \times_Y X$ and Nakayama's lemma. □

If $f : X \to Y$ and $g : Y \to Z$ are two morphisms, there is a canonical short exact sequence

$$f^*(\Omega^1_{Y/Z}) \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

The theorem therefore implies that if gf is unramified, then so is f. The definition of $\Omega^1_{X/Y}$ as the pullback $\Delta^*(\mathcal{J}/\mathcal{J}^2)$ (with obvious notation) allows us to conclude that if $X \to Y$ is a monomorphism (i.e: $X \to X \times_Y X$ is an isomorphism or, equivalently, $\mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y)$ is injective for all T), then $X \to Y$ is unramified. In particular, open and closed immersions (and inverse limits of such maps) are unramified.

The theorem also implies that the locus of ramification of a morphism $f : X \to Y$ is the closed subset which is the support of (the coherent sheaf) $\Omega^1_{X/Y}$. Thus, the set of points where a morphism is unramified form an open subset.

Subsection 3.3. **The functorial characterisation.** In basic algebraic geometry we learn that some classes of morphisms can be characterised functorially, and that such descriptions are incredibly useful. Unramified morphisms too have such a characterisation which we now present (assuming the morphism is locally of finite type).

Theorem 3.3.1. *Let $f : X \to S$ be a morphism that is locally of finite type. Then the following are equivalent:*

1. *f is unramified,*
2. *For all S-schemes $Y \to S$ which are affine, and subschemes Y_0 of Y defined by square-zero ideals, the natural map $Hom_S(Y, X) \to Hom_S(Y_0, X)$ is injective.*

Proof. Since both properties are local on the source and the target, we are free to assume that S and X are affine, say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. Thus, $Y = \mathrm{Spec}(C)$ is also affine. Let J be a square-zero ideal of C and assume that we are given the diagram

$$\begin{tikzcd} & B \arrow[d, "\phi"] \arrow[dr, "\bar{\phi}"] & \\ R \arrow[ur] \arrow[r] & C \arrow[r] & C/J \end{tikzcd}$$

One can easily verify that the association $\psi \to \psi - \phi$ gives a bijection between the set of liftings of $\bar{\phi}$ and the module $\mathrm{Der}_R(B, J)$. Thus, we obtain the implication (1) $\Rightarrow$ (2)

To obtain the reverse implication, consider the surjection $q : C = (B \otimes_R B)/I^2 \to B = C/J$ defined by the square zero ideal $J = I/I^2$ where I is the kernel of the multiplication map $B \otimes_R B \to B$. We already have a lifting $B \to C$ defined by, say, $b \mapsto b \otimes 1$. Thus, by the same reasoning as above, we obtain a bijective correspondence between liftings of $\mathrm{id} : B \to C/J$ and $\mathrm{Der}_R(B, J)$. The hypothesis therefore implies that the latter module is trivial. But we know that $J \cong \Omega^1_{B/R}$. Thus, B/R is unramified. □

Subsection 3.4. **Some topological properties.** The first topological result that will be of utility to us is one which says that unramified and separated morphisms have "nice" sections.

Proposition 3.4.1. *Any section of an unramified morphism is an open immersion, while any section of a separated morphism is a closed immersion. Thus, any section of an unramified separated morphism with a connected target is an isomorphism onto a connected component.*

Proof. Fix a base scheme S. If $g : X \to S$ is separated (resp. unramified) and $f : X' \to X$ is any S-morphism, then the graph $\Gamma_f : X' \to X' \times_S X$ is obtained as the base change of the diagonal $X \to X \times_S X$ via the projection $X' \times_S X \to X \times_S X$.

Since the diagonal is a closed immersion (resp. open immersion), so is the graph. In the special case $X' = S$, we obtain the claim. $\square$

We can now explicitly describe the sections of unramified morphisms.

Theorem 3.4.2. *If Y is a noetherian connected scheme and $f : X \to Y$ is unramified and separated, then every section of f is an isomorphism onto a connected component. There exists a bijective correspondence between sections of f and connected components X_i of X such that the induced map $X_i \to Y$ is an isomorphism. In particular, the knowledge of a section is equivalent to the knowledge of its value at any point in the base.*

Proof. Proposition 3.4.1 shows that a section of f has to be both an open and closed immersion and, consequently, it is an isomorphism onto its image. Therefore, it maps onto a connected component of Y. The rest follows easily. $\square$

The preceding theorem gives us some idea of the "rigidity" of unramified morphisms. Further indication is provided by the following proposition which, besides being intrinsically interesting, is also extremely useful in the theory of the algebraic fundamental group ([Gro71], exposé 5).

Proposition 3.4.3. *Let Y is a noetherian connected scheme, and $f : X \to Y$ be unramified and separated. Let $f, g : S \to X$ be two Y-morphisms such that $f(s) = g(s)$, and that the induced maps $\kappa(g(s)) = \kappa(f(s)) \to \kappa(s)$ are identical (that is, f and g are geometrically equal at x). Then $f = g$.*

Proof. The maps $f, g : S \to X$ defines the maps $(f, 1), (g, 1) : S \to X \times_Y S$. If we denote by $i : \operatorname{Spec}(\kappa(s)) \to S$ the canonical map from the residue class field at s, then the hypothesis ensures that $f \circ i = g \circ i$ and, consequently, $(f, 1) \circ i = (g, 1) \circ i$. Therefore, $(f, 1)(s) = (g, 1)(s)$. However, the maps $(f, 1)$ and $(g, 1)$ are sections of the unramified morphism $p_2 : X \times_Y S \to S$. Thus, by the preceeding theorem, since $(f, 1)$ and $(g, 1)$ agree geometrically at a point, they agree everywhere. $\square$

The topological results presented above will be used to give a functorial characterisation of étale morphisms similar to Theorem 3.3.1.

Subsection 3.5. **Examples.** We will end the section with a few examples.

Example 3.5.1 (The trivial case)**.** Unramified quasi-compact morphisms $X \to \operatorname{Spec}(k)$ for a field k are forced to be affine because X has to have dimension 0 and be compact. Noether normalisation (or whatever else you want) forces X to be the spectrum of a finite separable k-algebra A. Such algebras are simply products of finite separable field extensions of k. Thus, giving an unramified quasi-compact morphism to a field is not different from giving a finite number of separable field extensions of k. In particular, an unramified morphism with a connected source and a one point target is forced to be a finite separable field extensions. As we will see later, $X \to \operatorname{Spec}(k)$ is étale if and only if it is unramified. Thus, in this case at least, we obtain a very easy description of the étale topology of a scheme. Of course, the cohomology of this topology is another story.

Example 3.5.2 (The standard case)**.** Property 3 in 3.2.1 gives us a canonical source of examples for unramified morphisms. Fix a ring R and an integer n. Any ideal $J = (g_1, \cdots, g_m)$ in $R[x_1, \cdots, x_n]$ with the property that the matrix $(\frac{\partial g_i}{\partial x_j})$ has rank n at a point $x \in R^n$ defines a morphism $f : \mathrm{Spec}(R[x_1, \cdots, x_n]/J) \to \mathrm{Spec}(R)$ that is unramified at the point $x \in \mathbf{A}^n_R(R)$. Clearly we must have $m \geq n$. If we can choose $m = n$ (i.e: the differential of the map $\mathbf{A}^n_R \to \mathbf{A}^n_R$ defined by the g_i's is an isomorphism of the tangent spaces), a theorem of Grothendieck allows us to show that f is also flat x and, hence, is an étale map. Conversely, we will see that all étale maps arise locally in this manner.

Example 3.5.3 (Number theory)**.** Fix a Galois extension of number fields L/K with rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$. The injection $K \to L$ defines a morphism $f : \mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$. As discussed above, the points where f is unramified in our sense correspond to the set of points where f is unramified in the conventional sense. In the conventional sense, the locus of ramification in $\mathrm{Spec}(\mathcal{O}_L)$ can be defined by vanishing set of the "different" (this is an ideal in $\mathcal{O}_L$). (In fact, the different is nothing but the annihilator of $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$.) Similarly, the vanishing set of the discriminant (an ideal in $\mathcal{O}_K$) is precisely the set of points of K which ramify in L (that is, at least one prime lying above them is ramified). Thus, denoting by X the complement of the closed subset defined by the different in $\mathrm{Spec}(\mathcal{O}_L)$, and by Y the complement of the closed subset defined by the discriminant in $\mathrm{Spec}(\mathcal{O}_K)$, we obtain a morphism $X \to Y$ which is unramified. Furthermore, it is shown in algebraic number theory that this is also finite and flat. Thus, this is an example of an étale covering. The same situation of affairs can be mimcked for the function field case too.

Section 4. Flat morphisms

This section simply exists to summarise the properties of flatness that will be useful to us. Thus, we will be content with stating the theorems precisely and giving references for the proofs.

Subsection 4.1. **Definitions, sorites, and a theorem of Grothendieck.** After briefly recalling the necessary facts about flat modules over Noetherian rings, we state a theorem of Grothendieck which gives sufficient conditions for "hyperplane sections" of certain modules to be flat.

Definition 4.1.1. A module N over a ring A is said to be flat if the functor $M \to M \otimes_A N$ is exact. If this functor is also faithful, we say that N is faithfully flat over A. A morphism of rings $f : A \to B$ is said to be flat (resp. faithfully flat) if the functor $M \to M \otimes_A B$ is exact (resp. faithful and exact).

We first begin with some sorites, all of which can be found in [Mat70]. Clearly free and projective modules are flat. It's easily verified that flatness is a local property (that is, M is flat over A if and only if M_p is flat over A_p for all $p \in \mathrm{Spec}(A)$), and that finite flat modules over noetherian local rings are free. If $f : A \to B$ is a morphism of arbitrary rings, f is flat if and only if the induced maps $A_{f^{-1}q} \to B_q$ are flat for all $q \in \mathrm{Spec}(B)$. If $f : A \to B$ is a morphism of local rings, f is flat if and only if it is faithfully flat. Thus, a morphism of arbitrary rings is faithfully flat if and only if it is flat and the induced map on spectra is surjective. An important result from commutative algebra is that if A is a noetherian local ring, the completion $\widehat{A}$ is faithfully flat over A – this is the algebraic way of capturing the idea that "no local information is lost on passage to the completion." As a consequence of this, we obtain that a module M is flat over A if and only if $M \otimes_A \widehat{A}$ is flat over $\widehat{A}$ (that is, flatness can be checked after a base change to the completion). Before we move on to the geometric category, we present Grothendieck's theorem, which provides a convenient recipe for producing flat modules[1].

Theorem 4.1.2 (Grothendieck)**.** *Let $f : A \to B$ be a morphism of local rings. If M is a finite B-module that is flat as an A-module, and $t \in r(B)$ is an element such that multiplication by t is injective on $M/r(A)M$, then M/tM is also A-flat*

Proof. This essentially follows from the local flatness criterion of Grothendieck. The idea is to first prove that t is M-regular (i.e: multiplication by t is injective on M) and then give a Tor argument using the exact sequence $0 \to M \to M \to M/tM \to 0$ where the first map is multiplication by t. A carefully written out proof can be found, for instance, [Mat70], section 20. □

Definition 4.1.3. A morphism $f : X \to Y$ of schemes is said to be flat at $x \in X$ if the associated morphism of local rings at x ($\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$) is flat. The morphism $f : X \to Y$ is said to flat if it is flat at all points of x. A morphism $f : X \to Y$ that is flat and surjective is said to be faithfully flat.

Once again, some sorites are in order. The property (of a morphism) of being flat is, by fiat, local on the source and the target. Consequently, open immersions are flat. Almost as trivially, flat morphisms are stable under base change and composition. Slightly less trivially, $f : X \to Y$ is flat if and only if the functor f^* is exact on the category of quasi-coherent sheaves on Y.

Subsection 4.2. **Some topological properties.** We "recall" below some openness properties that flat morphisms enjoy.

Theorem 4.2.1. *For a morphism of finite type $f : X \to Y$, the set of points in X where f is flat is an open set. Moreover, if f is flat at all points of X, it is an open map. Thus, a flat morphism can be factored as a faithfully flat morphism followed by an open immersion.*

Proof. A proof of the first claim can be found in [DG67], IV, Section 11 or in [Gro71], Exposé IV, Section 6. The second claim depends on three results. The first one is Chevalley's theorem which states that f preserves constructible sets (this doesn't require flatness); the second one is the easy fact that constructible sets (of a noetherian scheme) are open if and only they are stable under generalisation; the last one is the fact that the "going-down" theorem holds for faithfully flat morphisms of rings. A proof of all three of these facts and how they imply the claim can be found in section 6 of [Mat70] □

Theorem 4.2.2. *A faithfully flat quasi-compact morphism is a quotient map for the Zariski topology.*

Proof. If $f : X \to Y$ is a surjective flat quasi-compact morphism, then f sends constructible sets to pro-constructible sets (application of Chevalley's theorem using the fact that, over a ring R, any R-algebra is a direct limit of finitely generated R-algebras). Such sets are closed if and only if they are stable under specialisation. Using this fact, the surjectivity of f, and the fact that the "going-down" theorem holds for faithfully flat morphisms of rings, one can easily show that f is a quotient

[1]We shall use this theorem later to give two equivalent definitions of smooth and étale morphisms.

map for the Zariski topology. Like the previous theorem, a proof of this theorem too can be found in section 6 of [Mat70]. □

An important reason to study flat morphisms is that they provide the adequate framework for capturing the notion of a family of schemes parametrised by the points of another scheme. Naively one may think that any morphism $f : X \to S$ should be thought of as a family parametrised by the points of S. However, without a flatness restriction on f, really bizarre things can happen in this so-called family. For instance, we aren't guaranteed that relative dimension (dimension of the fibres) is constant in a family. Other numerical invariants, such as the Hilbert polynomial, too may change from fibre to fibre. Flatness prevents such things from happening and, therefore, provides some "continuity" to the fibres.

Section 5. Étale morphisms

In this section, we will define étale morphisms and prove a number of important properties about them. The most important one, no doubt, is the functorial characterisation presented in Theorem 5.5.1. Following this, we will also discuss a few properties of rings which are insensitive to an étale extension (i.e: properties which hold for a ring if and only if they hold for all its étale extensions) to motivate the basic tenet of étale cohomology – étale morphisms are the algebraic analogue of local isomorphisms.

Subsection 5.1. **Definitions and sorites.** As the title suggests, we will define the class of étale morphisms – the class of morphisms (whose surjective families) we shall deem to be coverings in the category of schemes over a base scheme S in order to define the étale site S_{et}. Intuitively, an étale morphism is supposed to capture the idea of a covering space and, therefore, should be close to a local isomorphism. If we're working with varieties over algebraically closed fields, this last statement can be made into a definition provided we replace "local isomorphism" with "formal local isomorphism" (isomorphism after completion). One can then give a definition over any base field by asking that the base change to the algebraic closure be étale (in the aforementioned sense). But, rather than proceeding via such aesthetically displeasing constructions, we will adopt a cleaner, albeit slightly more abstract, algebraic approach.

Definition 5.1.1. A morphism $f : A \to B$ of local rings is étale if it is flat and unramified.

As we have already discussed the sorites for flat and unramified morphisms, there's not much more to discuss here. One thing that we would like to point out, however, is that étaleness can be checked after completion. Moreover, by combining flatness with basic properties of complete local rings, we see that if $f : A \to B$ is étale, then, in fact, $\widehat{B}$ is a finite flat $\widehat{A}$-module and, hence, $\widehat{B} \cong \left(\widehat{A}\right)^n$. The integer n is nothing other than the (separable) degree $[k(B) : k(A)]$. In particular, if $k(A)$ is separably closed, we obtain that $\widehat{A} \to \widehat{B}$ is an isomorphism, which vindicates our earlier claims. Lastly, if $f : A \to B$ is étale, the unramifiedness forces $\dim(B) \leq \dim(A)$ while (faithful) flatness forces the other inequality. Thus, we obtain that $\dim(B) = \dim(A)$.

Definition 5.1.2. A morphism $f : X \to Y$ of schemes is said to étale at $x \in X$ if it is flat and unramified at x (and, therefore, of finite type in a neighbourhood of x). The morphism is said to étale if it is étale at all its points.

Note that the unramifiedness hypothesis forces étale morphisms to be locally of finite type; flatness then forces such morphisms to be open. Since unramifiedness and flatness are both open properties, the étale locus of a morphism is open. Moreover, it's trivially verified that étaleness, besides being local on the source and the target, is stable under base change and composition.

Subsection 5.2. **The structure theorem for étale morphisms.** We present a theorem which describes the local structure of étale morphisms with great clarity. Besides its obvious independent importance, this theorem also allows us to make the transition to another definition of étale morphisms that captures the geometric intuition better than the one we've used so far.

Theorem 5.2.1 (Structure Theorem)**.** *Let $f : A \to B$ be an unramified morphism of local rings with the property that B is the localisation of a finitely generated A-algebra. Then there exists a finite A-algebra A', a maximal ideal $p \in A'$, a generator u of A' (as an A-algebra), a monic polynomial $F \in A[t]$ such that $F(u) = 0$ and $F'(u) \notin p$ and an isomorphism $B \to A'_p$ as A-algebras. Furthermore, we may choose $A' \cong A[t]/(F)$ if f is étale.*

Proof. The first step is to use Zariski's main theorem[2] to construct a finite A-algebra A' and a maximal ideal p of A' such that $A'_p \cong B$ as an A-algebra. The next step is to combine the primitive element theorem with Nakayama's lemma to be able to assume that A' is monogenic. The last step is to show that this A' works. A carefully written out proof can be found in section 7 of exposé 1 of [Gro71]. ☐

Via standard lifting arguments, one then obtains the following geometric statement which will be of essential use to us.

Theorem 5.2.2. *Let $f : X \to Y$ be an étale morphism. Then, for every $x \in X$, there exist affine neighbourhoods $V = Spec(R)$ and $U = Spec(S)$ of $f(x)$ and x respectively such that $f(U) \subset V$ and that U is V-isomorphic to an open subscheme of $Spec(R[t]/g)_{g'}$ for some monic polynomial $g \in R[t]$ (with $g' = dg/dt$ and that U is V-isomorphic to an open subscheme of $Spec(R[t]/g)_{g'}$ for some monic polynomial $g \in R[t]$ (with $g' = dg/dt$).*

Subsection 5.3. **An equivalent definition.** We now give another (equivalent) definition of étale morphisms which, besides having some geometric interpretation, is often easily verified in practice. More importantly perhaps, this definition also naturally leads one to the notion of smoothness. As smooth morphisms don't directly concern us, we don't discuss them here and, instead, refer the interested reader to chapter 2 of the Neron models book ([BLR90]) for an almost perfect account of the basic theory of smoothness, especially its relationship to differential calculus.

Definition 5.3.1. A morphism $f : X \to Y$ (of schemes) is said to be étale if the following two properties hold:

[2] The classical version, as explained in Section 4.4 of Chapter 1 of [DG67], III suffices for our purposes; we do not need the full power of Deligne's generalised version of the main theorem.

(1) for every $x \in X$, there exists an open neighbourhood U of x and an immersion $g : U \to \mathbf{A}^n_Y$, and
(2) if $\mathcal{J}$ is the sheaf of ideals that defines g, then, locally at $g(x)$, $\mathcal{J}$ can be generated by sections $g_1, \cdots, g_n$ such that the differentials dg_i form a basis for $\Omega^1_{\mathbf{A}^n_Y}$ at $g(x)$.

Proof of equivalence. Note that the first property simply expresses the fact that f is locally of finite type. Thus, étale morphisms for the old definition satisfy the first property. To show that they satisfy the second one as well, we use Theorem 5.2.2. Following the notation of that theorem, we may assume that $U = \mathrm{Spec} R[t, x, y]/(g, xg' - 1, ya - 1)$ where $V = \mathrm{Spec}(R)$ is an open subscheme of Y with $U \subset f^{-1}(V)$, g is a polynomial in t, $g' = dg/dt$ and a is a polynomial in t and x. It is then trivially verified that the obvious morphism $U \to \mathbf{A}^3_V \to \mathbf{A}^3_Y$ is an immersion with the requisite properties.

For the converse direction, let $f : X \to Y$ be a morphism verifying properties 1 and 2. By the first property, we get that f is locally of finite type. It remains to show that f is unramified and flat.

To see that f is unramified, using Theorem 3.2.1, it suffices to show that $\Omega^1_{X/Y} = 0$. Since this is a local statement, after fixing $x \in X$, we immediately reduce to the case where $Y = \mathrm{Spec}(R)$ and $g : X \to \mathbf{A}^n_Y$ is a closed subscheme defined by $J = (g_1, \cdots, g_n)$ with the property that the differentials dg_i form a basis for $\Omega^1_{\mathbf{A}^n_Y}$ at $g(x)$. We are now in a position to use the exact sequence

$$g^*(\mathcal{J}/\mathcal{J}^2) \to g^*(\Omega^1_{\mathbf{A}^n_Y/Y} \to \Omega^1_{X/Y} \to 0$$

where $\mathcal{J}$ is the sheaf of ideals associated to J. The hypothesis implies that the fibre of $\Omega^1_{X/Y}$ is 0 at x which implies that $\Omega^1_{X/Y}$ is trivial at x by Nakayama's lemma. Thus, we've shown that f is unramified.

To prove flatness, we once again reduce to the local case. Following the same notation as that introduced above, we need to show that $R[x_1, \cdots, x_n]/(g_1, \cdots, g_n)$ is flat over R where the x_i are coordinates on $\mathbf{A}^n_Y$ and $(\frac{\partial g_i}{\partial x_j})$ is invertible at $g(x)$. The flatness would follow from Grothendieck's flatness theorem (Theorem 4.1.2) if we showed that $g_1, \cdots, g_n$ was a $k(R)$-regular sequence inside $k(R)[x_1, \cdots, x_n]$. We know that $k(R)[x_1, \cdots, x_n]/(g_1, \cdots, g_n)$ is étale over $k(R)$ (we just showed it was unramified, and any morphism to a field is flat) and, consequently, of dimension 0. Hence, $\mathrm{ht}(g_1, \cdots, g_n) = n$ by basic dimension theory. Since $k(R)[x_1, \cdots, x_n]$ is a Cohen-Macaulay ring, it follows, from Theorem 17.4 in [Mat86] for instance, that $g_1, \cdots, g_n$ is a $k(R)$-regular sequence which finishes the proof. For a proof that avoids the use of Cohen-Macaulay rings, we refer the reader to Theorem 3 of Section 3.10 of Mumford's exposition ([Mum88]). □

Subsection 5.4. **Some topological properties.** We present a few of the fundamental topological properties of étale morphisms as explained in, say, [Gro71], exposé 1, section 5. Of key importance here is Theorem 5.4.3 which, besides providing one direction of the equivalence promised by the functorial characterisation, also gives motivation to view étaleness as essentially a topological property. But first, we give what Grothendieck calls the fundamental theorem for étale morphisms.

Theorem 5.4.1. *Let $f : X \to Y$ be a separated morphism of finite type. Then f is an open immersion if and only if it étale and radiciel*[3]*.*

Proof. It is clear that open immersions are étale and radiciel. For the converse direction, we are told that f is étale and universally injective. As it is a flat morphism of finite type, it also universally open. Hence, it is universally a homeomorphism onto its image. Replacing Y with $f(X)$, we may assume that $f : X \to Y$ is a universal homeomorphism. Now, if f had a section, the section would have to be an open immersion (because f is unramified) that is surjective (because f is a homeomorphism). That is, it would be an isomorphism and that would prove our claim. On the other hand, to show that f is an isomorphism, it clearly suffices to work after a faithfully flat base change. But f itself provides such a base change! And once we base change via f, the diagonal provides a section. The claim follows. □

Next, we present an extremely crucial theorem which, roughly speaking, says that étaleness is a topological property.

Theorem 5.4.2. *Let X and Y be two separated noetherian schemes over a base scheme S such that X is étale over S . Let S_0 be a subscheme of S defined by a nilpotent ideal, and denote by X_0 (resp. Y_0) the pullback $X \times_S S_0$ (resp. $Y \times_S S_0$). Then the map $Hom_S(Y, X) \to Hom_{S_0}(Y_0, X_0)$ is bijective.*

Proof. After base changing via $Y \to S$, we may assume that $Y = S$ in which case the theorem states that any Y-morphism $Y_0 \to X$ actually factors uniquely through a section $Y \to X$. For existence, assume that we are given $t : Y_0 \to X$. Since $|Y_0| = |Y|$, by Theorem 3.4.2, the section t is uniquely determined by a connected component X_i of X such that $X_i \times_Y Y_0 \to Y_0$ is an isomorphism (with inverse defined by (t, id)). In particular, $X_i \to Y$ is a universal homeomorphism and therefore radiciel. Since $X_i \to X$ and $X \to Y$ are étale, it follows from Theorem 5.4.1 that $X_i \to Y$ is an isomorphism and, therefore, it has an inverse which is the required section. The uniqueness follows from repeated application of Theorem 3.3.1, or directly from Theorem 3.4.2, or, if one carefully observes, from our proof itself. □

From the proof of preceeding theorem, we also obtain one direction of the promised functorial characterisation of étale morphisms.

Theorem 5.4.3. *Let $f : X \to S$ be an étale morphism. Then for all S-schemes $Y \to S$ which are affine, and subschemes Y_0 of Y defined by square-zero ideals, the natural map $Hom_S(Y, X) \to Hom_S(Y_0, X)$ is bijective.*

Subsection 5.5. **The functorial characterisation.** We finally present the promised functorial characterisation. Note that this takes our count of (equivalent) definitions of étale morphisms to four – the one we originally gave, the one provided by the structure theorem, the alternative one and the one obtained from the functorial characterisation.

Theorem 5.5.1. *Let $f : X \to S$ be a morphism that is locally of finite type. Then the following are equivalent*

[3]Recall ([DG67], I, chapter 1, section 3.5) that $f : X \to Y$ is radiciel if $X(K) \to Y(K)$ is injective for every field K, and that this is equivalent to requiring that f be injective and that the maps $\kappa(f(x)) \to \kappa(x)$ be epimorphisms in the category of fields (purely inseparable extensions). Lastly, this is also equivalent to requiring that f be universally injective

(1) *f is étale*
(2) *For all S-schemes $Y \to S$ which are affine, and subschemes Y_0 of Y defined by square-zero ideals, the natural map $Hom_S(Y, X) \to Hom_S(Y_0, X)$ is bijective.*

Proof. The forward implication was proven in Theorem 5.4.3. For the reverse implication, we use Definition 5.3.1. We may assume that X is defined as a closed subscheme $g : X \to \mathbf{A}^n_S$ by an ideal $\mathcal{J}$. Using the alternative definition, it suffices to show that the natural map $g^*(\mathcal{J}/\mathcal{J}^2) \to g^*(\Omega^1_{\mathbf{A}^n_S/S})$ is an isomorphism. Since this is a local problem, we may assume that $S = \operatorname{Spec}(R)$, $\mathbf{A}^n_S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ where $A = R[x_1, \cdots, x_n]$ and B is a quotient of A by an ideal I. We have the canonical isomorphism $B \to (A/I^2)/(I/I^2)$ which, by the functorial hypothesis, lifts to an R-linear map $B \to A/I^2$. Therefore, the exact sequence $0 \to I/I^2 \to A/I^2 \to A/I \to 0$ splits. If we denote the first map by i, the second map by v and the splitting $A/I \to A/I^2$ by ϕ, then $\tau = \mathrm{id} - (\phi \circ v)$ defines an A-derivation $A/I^2 \to I/I^2$. Consequently, we obtain a map $\Omega^1_{A/R} \otimes_A B \to I/I^2$ which gives an inverse to the natural map $I/I^2 \to \Omega^1_{A/R} \otimes_A B$ thereby showing that the latter is an isomorphism, as was required. □

This characterisation says that solutions to the equations defining X can be lifted uniquely through nilpotent thickenings.

Subsection 5.6. **Permanence properties.** We have already seen that the Krull dimension is insensitive to an étale extension. In what follows, we present a few other such "permanence" properties of étale morphisms.

Proposition 5.6.1. *Let $f : A \to B$ be an étale map of local rings. Then* $\operatorname{depth}(A) = \operatorname{depth}(B)$

Proof. This follows fairly easily from the observation that, on tensoring with B, the Koszul complex of the ideal $r(A)$ of A gives the Koszul complex of the ideal $r(B)$ of B, and that $A \to B$ is faithfully flat. □

Proposition 5.6.2. *Let $f : A \to B$ be an étale map of local rings. Then A is regular if and only if B is so.*

Proof. By the étaleness of $A \to B$ and the local flatness criterion ([Mat70], theorem 49), one sees that $gr^*(B) \cong gr^*(A) \otimes_{k(A)} k(B)$ as graded algebras. Thus, by looking at the degree 1 components, we see that the embedded dimensions of A and B co-incide. By the étaleness of $A \to B$, the (Krull) dimensions of the two rings co-incide as well. Thus, A is regular if and only if B is so. □

Proposition 5.6.3. *Let $f : A \to B$ be an étale map of local rings. Then A is reduced if and only if B is so.*

Proof. It's clear from the faithful flatness of $A \to B$ that if B is reduced, so is A. Conversely, lets assume A is reduced and show that B is so. By assumption, if $\{p_i\}$ is the set of minimal primes of A, the natural map $A \to \prod_i A/p_i$ is injective. By the flatness of B, $B \to \prod_i B/p_iB$ is also injective; hence, it suffices to show that each of B/p_iB is reduced. Thus, after base changing to an irreducible component, we may assume that A is a domain with field of fractions K. By the flatness of B, the natural map $B \to B \otimes_A K$ is injective; hence, it suffices to show the latter is reduced. Since $K \to B \otimes_A K$ is étale, we are reduced to the case where A is a field. By virtue of Example 3.5.1, we see that B is a product of fields, and therefore reduced. □

Proposition 5.6.4. *Let $f : A \to B$ be an étale map of local rings. Then A is normal if and only if B is so.*

Proof. We use Serre's normality criterion for a noetherian local ring A of dimension $\neq 0$. Recall that this says that A is normal if and only if it is regular in codimension 1, and for every prime p of height ≥ 2, $\mathrm{depth}(A_p) \geq 2$. Since $A \to B$ is an étale map of local rings, it's faithfully flat. Moreover, if $p \in \mathrm{Spec}(B)$ lies over $q \in \mathrm{Spec}(A)$, then $A_q \to B_p$ is étale. Hence, the height 1 (resp. ≥ 2) primes of B lie over all the height 1 (resp. ≥ 2) primes of A. The result now follows from the permanence of regularity and depth for étale extensions. □

Proposition 5.6.5. *Let $f : A \to B$ be an étale map of local rings. Then A is Cohen-Macaulay if and only if B is so.*

Proof. Recall that a local ring A is Cohen-Macaulay if and only $\dim(A) = \mathrm{depth}(A)$. As each of these invariants is preserved under an étale extension, the claim follows. □

The preceeding propositions give some indication as to why we'd like to think of étale maps as "local isomorphisms". Another property that gives an excellent indication that we have the "right" definition is the fact that for $\mathbf{C}$-schemes of finite type, a morphism is étale if and only if the associated morphism on analytic spaces (the $\mathbf{C}$-valued points given the complex topology) is a local isomorphism in the analytic sense (open embedding locally on the source). This fact can be proven with the aid of the structure theorem and the fact that the analytification commutes with the formation of the completed local rings – the details are left to the reader.

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(2) go back to the table of contents: index.html#contents.

References

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron Models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1990.
[DG67] Jean Dieudonné and Alexandre Grothendieck. Éléments de géométrie algébrique. *Inst. Hautes Études Sci. Publ. Math.*, 4, 8, 11, 17, 20, 24, 28, 32, 1961-1967.
[Gro71] Alexandre Grothendieck. *Revêtements etales et groupe fondamental (SGA 1)*, volume 224 of *Lecture notes in mathematics*. Springer-Verlag, 1971.
[Mat70] Hideyuki Matsumura. *Commutative Algebra*. W. A. Benjamin, Inc., 1970.
[Mat86] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1986.
[Mum88] David Mumford. *The Red Book of Varieties and Schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, 1988.