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CATEGORIES

Contents

Section 1. Introduction

Categories were first introduced in [EL45].

Section 2. Categories and 2-categories

The category of categories (which does not exist with our conventions) is a 2-category. Similarly, the category of stacks forms a 2-category. So, even if you already know about categories, you can read this section and find the terminology regarding 1-morphisms and 2-morphisms that we will use later for stacks as well.

Subsection 2.1. **Categories.** We recall the definitions, partly to fix notation. A category $\mathcal{C}$ consists of the following data:

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a set of morphisms $\mathrm{Mor}_\mathcal{C}(x, y)$.
(3) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a composition map $\mathrm{Mor}_\mathcal{C}(y, z)\times\mathrm{Mor}_\mathcal{C}(x, y) \to \mathrm{Mor}_\mathcal{C}(x, z)$, denoted $(\phi, \psi) \mapsto \phi \circ \psi$.

These data are to satisfy the following rules:

(1) For every element $x \in \mathrm{Ob}(\mathcal{C})$ there exists a unique identity morphism $\mathrm{id}_x \in \mathrm{Mor}_\mathcal{C}(x, x)$ such that $\mathrm{id}_x \circ \phi = \phi$ and $\psi \circ \mathrm{id}_x = \psi$ whenever these compositions make sense.
(2) Composition is transitive $(\phi\circ\psi)\circ\chi = \phi\circ(\psi\circ\chi)$ whenever these compositions make sense.

We leave it to the reader to define the notion of an isomorphism between objects of the category $\mathcal{C}$.

Definition 2.1.1. A groupoid is a category where every morphism is an isomorphism.

Example 2.1.2. A group G can be thought of as a groupoid with a single object x and morphisms $\mathrm{Mor}(x, x) = G$, with the composition rule given by the group law in G.

Example 2.1.3. Any set C the set of objects of a groupoid $\mathcal{C}$ if we let $\mathrm{Ob}(\mathcal{C}) = C$ and declare $\mathrm{Mor}(x, y)$ to be empty if $x \neq y$ and to be $\{\mathrm{id}_x\}$ if $x = y$.

A functor $F : \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}, \mathcal{B}$ is given by the following data:

(1) A map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$.
(2) For every $x, y \in \mathrm{Ob}(\mathcal{A})$ a map $F : \mathrm{Mor}_\mathcal{A}(x, y) \to \mathrm{Mor}_\mathcal{B}(F(x), F(y))$, denoted $\phi \mapsto F(\phi)$.

These data should be compatible with composition and identity morphisms in an obvious manner. Note that every category $\mathcal{A}$ has an identity functor $\mathrm{id}_\mathcal{A}$.

Remark 2.1.4. Suppose that $\mathcal{A}$ is a category. A functor F from $\mathcal{C}$ to Sets clearly makes sense even though there is no category of sets. For example we can think of F as a functor into Sets_α, see Sets, Section 3.

Recall that a functor $F : \mathcal{A} \to \mathcal{B}$ is faithfull if for any objects x, y of $\mathrm{Ob}(\mathcal{A})$ the map $F : \mathrm{Mor}_\mathcal{A}(x, y) \to \mathrm{Mor}_\mathcal{B}(F(x), F(y))$ is injective. Moreover, if these maps are all bijective then F is called fully faithfull. The functor F is called essentially surjective if for any object $z \in \mathrm{Ob}(\mathcal{B})$ there exists an object $x \in \mathrm{Ob}(\mathcal{A})$ such that $F(x)$ is isomorphic to z in $\mathcal{B}$.

Example 2.1.5. A homomorphism $p \colon G \to H$ of groups gives rise to a functor between the associated groupoids in Example 2.1.2. It is faithful (resp. fully faithful) if and only if p is injective (resp. an isomorphism).

Example 2.1.6. Given a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$ we define the category of objects over X, denoted $\mathcal{C}/X$ as follows. The objects of $\mathcal{C}/X$ are morphisms $Y \to X$ for some $Y \in \mathrm{Ob}(\mathcal{C})$. Morphisms between objects $Y \to X$ and $Y' \to X$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X \colon \mathcal{C}/X \to \mathcal{C}$ which simply forgets the morphism for X. Moreover given a morphism $f \colon X' \to X$ in $\mathcal{C}$ there is an induced functor $F \colon \mathcal{C}/X' \to \mathcal{C}/X$ obtained by composition with f, and $p_X \circ F = p_{X'}$.

A transformation of functors $t : F \to G$ (or simply a morphism of functors) between functors $F, G : \mathcal{A} \to \mathcal{B}$ is given by the following data:

(1) For every $x \in \mathrm{Ob}(\mathcal{A})$ a morphism $t_x : F(x) \to G(x)$ in the category $\mathcal{B}$.

These data should satisfy the condition that for every morphism $\phi : x \to y$ of $\mathcal{A}$ the following diagram is commutative

$$\begin{tikzcd} F(x) \arrow[r, "t_x"] \arrow[d, "F(\phi)"'] & G(x) \arrow[d, "G(\phi)"] \\ F(y) \arrow[r, "t_y"] & G(y) \end{tikzcd}$$

Note that every functor F comes with the identity transformation $\mathrm{id}_F : F \to F$.

Next we recall the concept of equivalent categories. An equivalence of categories $F : \mathcal{A} \to \mathcal{B}$ is a functor such that there exists a functor $G : \mathcal{B} \to \mathcal{A}$ with the property that the compositions $F \circ G$ and $G \circ F$ are isomorphic to the identity functors $\mathrm{id}_\mathcal{B}$ respectively $\mathrm{id}_\mathcal{A}$.

Since we are working with categories that are sets it is easy to see that a functor is an equivalence of categories if and only if it is both fully faithful and essentially surjective.

Subsubsection 2.1.1. *Additional notions.*

Definition 2.1.7. Let $x, y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_\mathcal{C}(x, z)$ and $g \in \mathrm{Mor}_\mathcal{C}(y, z)$. The fibre product of f and g is an object $x \times_z y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_\mathcal{C}(x \times_z y, x)$ and $q \in \mathrm{Mor}_\mathcal{C}(x \times_z y, y)$ making the diagram

$$\begin{tikzcd}
x \times_y z \arrow[r, "p"] \arrow[d, "q"'] & x \arrow[d, "f"] \\
y \arrow[r, "g"] & z
\end{tikzcd}$$

commute, and such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_\mathcal{C}(w, x)$ and $\beta \in \mathrm{Mor}_\mathcal{C}(w, y)$ with $f \circ \alpha = g \circ \beta$ there is a unique $\gamma \in \mathrm{Mor}_\mathcal{C}(w, x \times_z y)$ making the diagram

$$\begin{tikzcd}
w \arrow[drr, "\alpha"] \arrow[dr, dashed, "\gamma"] \arrow[ddr, "\beta"'] & & \\
& x \times_y z \arrow[r, "p"'] \arrow[d, "q"'] & x \arrow[d, "f"] \\
& y \arrow[r, "g"] & z
\end{tikzcd}$$

commute. If a fibre product exists it is unique up to unique isomorphism. We say the category $\mathcal{C}$ has fibre products if the fibre product exists for any $f \in \mathrm{Mor}_\mathcal{C}(x, z)$ and $g \in \mathrm{Mor}_\mathcal{C}(y, z)$.

Given a category $\mathcal{C}$ we can form the opposite category $\mathcal{C}^{\mathrm{opp}}$ which has the same objects as $\mathcal{C}$ but all morphisms reversed, so $\mathrm{Mor}_{\mathcal{C}^{\mathrm{opp}}}(x, y) = \mathrm{Mor}_\mathcal{C}(y, x)$. A contravariant functor $F\colon \mathcal{C} \to \mathcal{S}$ is a functor $\mathcal{C}^{\mathrm{opp}} \to \mathcal{S}$. Concretely, if F is contravariant then for composable morphisms f and g in $\mathcal{C}$, $F(f \circ g) = F(g) \circ F(f)$.

Example 2.1.8. For any $U \in \mathrm{Ob}(\mathcal{C})$ there is a contravariant functor

$$\mathrm{Mor}(-, U)\colon \mathcal{C} \to \mathrm{Sets}$$

which takes an object X to the set $\mathrm{Mor}_\mathcal{C}(X, U)$. Given a morphism $f\colon X \to Y$ the corresponding map $\mathrm{Mor}(-, U)(f)\colon \mathrm{Mor}(Y, U) \to \mathrm{Mor}(X, U)$ takes ϕ to $\phi \circ f$. More commonly this functor is denoted $h_U : \mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets}$. If $\mathcal{C}$ is the category of schemes this functor is sometimes referred to as the *functor of points* of U.

Definition 2.1.9. A contravariant functor $F\colon \mathcal{C} \to \mathrm{Sets}$ is said to be representable if it is isomorphic to the functor $h_U(-) = \mathrm{Mor}(-, U)$ for some object U of $\mathcal{C}$.

Definition 2.1.10. A morphism $f : x \to y$ of a category $\mathcal{C}$ is said to be representable, if and only if for every morphism $z \to y$ in $\mathcal{C}$ the fibre product $z \times_y x$ exists.

Subsection 2.2. **2-categories.** We will give a definition of (strict) 2-categories as they appear in the setting of stacks. Before you read this take a look at Example 2.2.2. Basically, you take this example and you write out all the rules satisfied by the objects, 1-morphisms and 2-morphisms in that example. This is actually not that helpful but it shows that it can be done.

Definition 2.2.1. A 2-category $\mathcal{C}$ consists of the following data

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a set of 1-morphisms $\mathrm{Mor}_\mathcal{C}(x, y)$.
(3) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a composition map $\mathrm{Mor}_\mathcal{C}(y, z) \times \mathrm{Mor}_\mathcal{C}(x, y) \to \mathrm{Mor}_\mathcal{C}(x, z)$, denoted $(F, G) \mapsto F \circ G$.
(4) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ and for each pair $F, F' \in \mathrm{Mor}_\mathcal{C}(x, y)$ a set of 2-morphisms $\mathrm{Mor}_\mathcal{C}(F, F')$.
(5) For each triple F, F', F'' of 1-morphisms with the same source and target a composition law $\mathrm{Mor}_\mathcal{C}(F', F'') \times \mathrm{Mor}_\mathcal{C}(F, F') \to \mathrm{Mor}_\mathcal{C}(F, F'')$, denoted $(\phi, \psi) \mapsto \phi \circ \psi$.
(6) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and 1-morphisms $F, F' : x \to y$ and $G : y \to z$ a map $G : \mathrm{Mor}_\mathcal{C}(F, F') \to \mathrm{Mor}_\mathcal{C}(G \circ F, G \circ F')$.
(7) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and 1-morphisms $F : x \to y$ and $G, G' : y \to z$ a map $F : \mathrm{Mor}_\mathcal{C}(G, G') \to \mathrm{Mor}_\mathcal{C}(G \circ F, G' \circ F)$.

These data are to satisfy the following rules:

(1) For every element $x \in \mathrm{Ob}(\mathcal{C})$ there exists a unique identity 1-morphism $\mathrm{id}_x \in \mathrm{Mor}_\mathcal{C}(x, x)$ such that $\mathrm{id}_x \circ F = F$ and $G \circ \mathrm{id}_x = G$ whenever these compositions make sense.
(2) For every 1-morphism F there exists a unique identity 2-morphism $\mathrm{id}_F \in \mathrm{Mor}_\mathcal{C}(F, F)$ such that $\mathrm{id}_F \circ \phi = \phi$ and $\psi \circ \mathrm{id}_F = \psi$ whenever these compositions make sense.
(3) Composition is transitive for both 1-morphisms and 2-morphisms.
(4) Every 2-morphism is an isomorphism. This makes sense since the conditions sofar imply that $\mathrm{Mor}_\mathcal{C}(x, y)$ is a category with 1-morphisms as objects and 2-morphisms as morphisms. So this condition means every $\mathrm{Mor}_\mathcal{C}(x, y)$ is a groupoid.
(5) Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and let $G \in \mathrm{Mor}_\mathcal{C}(y, z)$. The map $\mathrm{Mor}_\mathcal{C}(x, y) \to \mathrm{Mor}_\mathcal{C}(x, z)$ given by G (see item 6 above) is a functor.
(6) Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and let $F \in \mathrm{Mor}(x, y)$. The map $\mathrm{Mor}_\mathcal{C}(y, z) \to \mathrm{Mor}_\mathcal{C}(x, z)$ given by F (see item 7 above) is a functor.
(7) Suppose we have objects x, y, z, 1-morphisms $F, F' : x \to y$, $G, G' : y \to z$, and 2-morphisms $\phi : F \to F'$, $\psi : G \to G'$. The following diagram commutes:

$$\begin{tikzcd}
G \circ F \arrow[r, "G(\phi)"] \arrow[d, "F(\psi)"'] & G \circ F' \arrow[d, "F'(\psi)"] \\
G \circ F' \arrow[r, "G'(\phi)"'] & G' \circ F'
\end{tikzcd}$$

This is obviously not a very pleasant type of object to work with. On the other hand, there are lots of examples where it is quite clear how you work with it. Note that we require the 2-morphisms to be isomorphisms. As far as this text is concerned all 2-categories occuring in this document are (full) sub 2-categories of the example below. FIXME: Remove this definition? Replace by a better one?

Example 2.2.2. The category of categories. The terminology of 2-categories applies to categories as follows. Choose an ordinal α (see our discussion in Sets, Section 3). Let $\mathrm{Ob}(\mathrm{Cat}_\alpha)$ be the set of all categories which are elements of V_α. This will be the set of objects of our 2-category. A 1-morphism between $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathrm{Cat}_\alpha)$ is a functor $F : \mathcal{A} \to \mathcal{B}$. A 2-morphism is an *isomorphism* of 1-morphisms, i.e., an invertible natural transformation of functors.

Composition of functors and composition of transformations of functors is defined above. Datum (6) in Definition 2.2.1 is given as follows. Suppose that $t : F \to F'$ is a transformation of functors $\mathcal{A} \to \mathcal{B}$ and suppose that $G : \mathcal{B} \to \mathcal{C}$ is a functor. In this case $G(t)$ is the transformation of functors $G \circ F \to G \circ F'$ given by $G(t_A) : G(F(A)) \to G(F'(A))$. Datum (7) of Definition 2.2.1 is defined similarly. FIXME. Check the rules (1) – (7) hold in this example (and no more than that in general).

The notion of equivalence of categories that we defined in Subsection 2.1 extends to the more general setting of 2-categories as follows.

Definition 2.2.3. Two objects x, y of a 2-category are *equivalent* if there exist 1-morphisms $F : x \to y$ and $G : y \to x$ such that $F \circ G$ is 2-isomorphic to id_y and $G \circ F$ is 2-isomorphic to id_x.

Remark 2.2.4. There are variants of the construction of 2.2.2 above where we look at the 2-category of groupoids (contained in some α), or categories fibred in groupoids over a fixed category, or stacks. And so on.

Remarks 2.2.5. (1) A functor from an ordinary category into a 2-category will ignore the 2-morphisms unless mentioned otherwise. In other words, it will be a "usual" functor into the category formed out of 2-category by forgetting all the 2-morpshisms.

(2) Another notion of a functor from a category $\mathcal{A}$ into a 2-category $\mathcal{C}$ would be to say that it is given by a map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{C})$ together with a family of maps $F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(F(x), F(y))$ such that for every composable pair of morphisms f, g of $\mathcal{A}$ the morphisms $F(g \circ f)$ and $F(g) \circ F(f)$ are 2-isomorphic. This is not a very good notion, since for example it does not require $F(\mathrm{id}_x)$ to be isomorphic to $\mathrm{id}_{F(x)}$. Even if you do then there may be a problem: see the conditions in (3) below.

(3) A better notion is the following. A weak functor (or a pseudo-functor) from a category $\mathcal{A}$ into a 2-category $\mathcal{C}$ is given by

(1) a map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{C})$,
(2) for every pair $x, y \in \mathrm{Ob}(\mathcal{A})$ a map $F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(F(x), F(y))$,
(3) for every $x \in \mathrm{ob}(C)$ a 2-morphism $\alpha_x : \mathrm{id}_x \to F(\mathrm{id}_x)$, and
(4) for every pair of composable morphisms f, g of $\mathcal{A}$ a 2-morphism $\alpha_{f,g} : F(g \circ f) \to F(g) \circ F(f)$.

Now these data are subject to the following conditions: (with notations as in Definition 2.2.1)

(1) for any morphism $f : x \to y$ in $\mathcal{A}$ the morphism $\alpha_{f,\mathrm{id}_y} : F(f) \to F(f) \circ F(\mathrm{id}_y)$ equals the composition of $F(f) \circ \mathrm{id}_{F(y)} = F(f)$ with $F(f)(\alpha_y)$, and similary for $\alpha_{\mathrm{id}_x,f}$ and α_x, and

(2) for any triple of composable morphisms f, g, h the compositions $F(h)(\alpha_{f,g}) \circ \alpha_{g\circ f,h}$ and $F(f)(\alpha_{g,h}) \circ \alpha_{g,f\circ h}$ should be equal.

Again this is not a very workable notion, but it does sometimes come up. There is a theorem that says that any pseudo-functor is isomorphic to a functor. FIXME: Add more as needed.

Subsubsection 2.2.1. *2-fibre products.* In this subsection we introduce 2-fibre products. Suppose that $\mathcal{C}$ is a 2-category. We say that a diagram

$$\begin{tikzcd} w \arrow[r] \arrow[d] & y \arrow[d] \\ x \arrow[r] & z \end{tikzcd}$$

2-commutes if the two 1-morphisms $w \to y \to z$ and $w \to x \to z$ are 2-isomorphic. In a 2-category it is more natural to ask for 2-commutativity of diagrams than for actually commuting diagrams. (Indeed, some may say that we should not work with strict 2-categories at all, and in a "weak" 2-category the notion of a commutative diagram of 1-morphisms does not even make sense.) Correspondingly the notion of a fibre product has to be adjusted.

Let $\mathcal{C}$ be a 2-category. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_{\mathcal{C}}(x, z)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(y, z)$. In order to define the 2-fibre product of f and g we are going to look at 2-commutative diagrams

$$\begin{tikzcd} w \arrow[r, "a"] \arrow[d, "b"'] & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z. \end{tikzcd}$$

Now in the case of categories, the fibre product is a final object in the category of such diagrams. Correspondingly a 2-fibre product is a final object in a 2-category (see definition below). The 2-category we will consider is the 2-category of 2-commutative diagrams defined as follows:

(1) Objects are quadruples (w, a, b, ϕ) as above where ϕ is a 2-morphism $\phi : f \circ a \to g \circ b$,
(2) 1-morphisms from (w, a, b, ϕ) to (w', a', b', ϕ') are given by $(k : w \to w', \alpha : a' \to a \circ k, \beta : b \circ k \to b')$ such that ϕ' equals

$$f \circ a' \xrightarrow{f(\alpha)} f \circ a \circ k \xrightarrow{k(\phi)} g \circ b \circ k \xrightarrow{g(\beta)} g \circ b'.$$

(3) a 2-morphism between (k_i, α_i, β_i), $i = 1, 2$ is given by a 2-morphism $\delta : k_1 \to k_2$ such that

$$\begin{tikzcd} a' \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_2"'] & a \circ k_1 \arrow[d, "a(\delta)"] \\ & a \circ k_2 \end{tikzcd} \qquad \begin{tikzcd} b \circ k_1 \arrow[r, "\beta_1"] \arrow[d, "b(\delta)"'] & b' \\ b \circ k_2 \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

commute.

Definition 2.2.6. A final object of a 2-category $\mathcal{C}$ is an object x such that (1) for every $y \in \mathrm{Ob}(\mathcal{C})$ there is a morphism $y \to x$, and (2) every two morphisms $y \to x$ are isomorphic by a unique 2-morphism.

Definition 2.2.7. Let $\mathcal{C}$ be a 2-category. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_\mathcal{C}(x, z)$ and $g \in \mathrm{Mor}_\mathcal{C}(y, z)$. A 2-fibre product of f and g is a final object in the category of 2-commutative diagrams described above. If a 2-fibre product exists we will denote it $x \times_z y \in \mathrm{Ob}(\mathcal{C})$, and denote the required morphisms $p \in \mathrm{Mor}_\mathcal{C}(x \times_z y, x)$ and $q \in \mathrm{Mor}_\mathcal{C}(x \times_z y, y)$ making the diagram

$$\begin{tikzcd} x \times_y z \arrow[r, "p"] \arrow[d, "q"'] & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z \end{tikzcd}$$

2-commute and we will denote the given 2-morphism exhibiting this by $\psi : f \circ p \to g \circ q$.

Thus the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $a \in \mathrm{Mor}_\mathcal{C}(w, x)$ and $b \in \mathrm{Mor}_\mathcal{C}(w, y)$ with a given 2-morphism $\phi : f \circ a \to g \circ b$ there is a $\gamma \in \mathrm{Mor}_\mathcal{C}(w, x \times_z y)$ making the diagram

$$\begin{tikzcd} w \arrow[dr, dashed, "\gamma"] \arrow[drr, "a"] \arrow[ddr, "b"'] & & \\ & x \times_y z \arrow[r, "p"] \arrow[d, "q"'] & x \arrow[d, "f"] \\ & y \arrow[r, "g"] & z \end{tikzcd}$$

2-commute such that for suitable choices of $q \circ \gamma \to b$ and $a \to p \circ \gamma$ the composition

$$f \circ a \longrightarrow f \circ p \circ \gamma \xrightarrow{\gamma(\psi)} g \circ q \circ \gamma \longrightarrow g \circ b$$

equals ϕ. Of course the exact properties are finer than this. All of the cases of 2-fibre products that we will need later on come from the following example of 2-fibre products in the 2-category of categories.

Example 2.2.8. In this example we switch notations and we let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories and we let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. In this case the 2-fibre product $\mathcal{A} \times_\mathcal{C} \mathcal{B}$ exists and is given by the following:

(1) an object of $\mathcal{A} \times_\mathcal{C} \mathcal{B}$ is a triple (A, B, f), where $A \in \mathrm{Ob}(\mathcal{A})$, $B \in \mathrm{Ob}(\mathcal{B})$, and $f : F(A) \to G(B)$ is an isomorphism in $\mathcal{C}$,

(2) a morphism $(A, B, f) \to (A', B', f')$ is given by a pair (a, b), where $a : A \to A'$ is a morphism in $\mathcal{A}$, and $b : B \to B'$ is a morphism in $\mathcal{B}$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "f"] \arrow[d, "F(a)"] & G(B) \arrow[d, "G(b)"] \\ F(A') \arrow[r, "f'"] & G(B') \end{tikzcd}$$

is commutative.

The functors $p : \mathcal{A} \times_\mathcal{C} \mathcal{B} \to \mathcal{A}$ and $q : \mathcal{A} \times_\mathcal{C} \mathcal{B} \to \mathcal{B}$ are the forgetfull functors in this case. The transformation $\psi : F \circ p \to G \circ q$ is given on the object $\xi = (A, B, f)$ by $\psi_\xi = f : F(p(\xi)) = F(A) \to G(B) = G(q(\xi))$.

Let us check the universal property: let $\mathcal{W}$ be a category, let $X : \mathcal{W} \to \mathcal{A}$ and $Y : \mathcal{W} \to \mathcal{B}$ be functors, and let $t : F \circ X \to G \circ Y$ be an isomorphism of functors. The desired functor $\gamma : \mathcal{W} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is given by $W \mapsto (X(W), Y(W), t_W)$. What else could it be? (A meta-argument for uniqueness.) FIXME: write this out.

Note that the functor γ constructed above actually has the property that $p \circ \gamma = X$ and $q \circ \gamma = Y$. In general this need not be the case.

Section 3. Categories fibred in groupoids

In this section we explain how to think about categories in groupoids and we see how they are basically the same as functors in groupoids.

Subsection 3.1. Definitions.

In this subsection we have a functor $p : \mathcal{S} \to \mathcal{C}$. We think of $\mathcal{S}$ as being on top and of $\mathcal{C}$ as being at the bottom.

Analogously to the fibre of a map of spaces, we have the notion of a fibre category. The fibre category over an object $U \in \mathrm{Ob}(\mathcal{C})$ is the category $\mathcal{S}_U$ with objects

$$\mathrm{Ob}(\mathcal{S}_U) = \{x \in \mathrm{Ob}(\mathcal{S}) : p(x) = U\}$$

and morphisms

$$\mathrm{Mor}_{\mathcal{S}_U}(x, y) = \{\phi \in \mathrm{Mor}_{\mathcal{S}}(x, y) : p(\phi) = \mathrm{id}_U\}.$$

In order to discuss the notion of "category fibred in groupoids" we temporarily introduce the notion of lifting. A *lift* of an object $U \in \mathrm{Ob}(\mathcal{C})$ is an object $x \in \mathrm{Ob}(\mathcal{S})$ such that $p(x) = U$, i.e., $x \in \mathrm{Ob}(\mathcal{S}_U)$. Similarly, a *lift* of a morphism $f : V \to U$ in $\mathcal{C}$ is a morphism $\phi : y \to x$ in $\mathcal{S}$ such that $p(\phi) = f$.

Definition 3.1.1. We say that $\mathcal{S}$ is fibred in groupoids over $\mathcal{C}$ if the following two conditions hold:

(1) For every morphism $f : V \to U$ in $\mathcal{C}$ and every lift x of U there is a lift $\phi : y \to x$ of f with target x.
(2) For every pair of morphisms $\phi : y \to x$ and $\psi : z \to x$ and any morphism $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ there exists a unique lift $\chi : z \to y$ of f such that $\phi \circ \chi = \psi$.

Condition (2) phrased differently says that applying the functor p gives a bijection between the sets of dotted arrows in the following commutative diagram below:

$$\begin{tikzcd} y \arrow[r] & x \\ z \arrow[u, dashed] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} p(y) \arrow[r] & p(x) \\ p(z) \arrow[u, dashed] \arrow[ur] & \end{tikzcd}$$

Another way to think about the second condition is the following. Suppose that $g : W \to V$ and $f : V \to U$ are morphisms in $\mathcal{C}$. Let $x \in \mathrm{Ob}(\mathcal{S}_U)$. By the first condition we can lift f to $\phi : y \to x$ and then we can lift g to $\psi : z \to y$. Instead of doing this two step process we can directly lift $g \circ f$ to $\gamma : z' \to x$. This gives the solid arrows in the diagram below.

$$\begin{tikzcd}
z' \arrow[drr, "\gamma"] \arrow[d, dashed, <->] & & \\
z \arrow[r, "\psi"] & y \arrow[r, "\phi"] & x
\end{tikzcd}$$

$$W \xrightarrow{g} V \xrightarrow{f} U$$

Applying the second condition to the arrows $\phi \circ \phi$, γ and id_W we conclude that there is a unique morphism $\chi : z \to z'$ in $\mathcal{S}_W$ such that $\gamma \circ \chi = \phi \circ \psi$. Similarly there is a unique morphism $z' \to z$. The uniqueness implies that the morphisms $z' \to z$ and $z \to z'$ are mutually inverse, in other words isomorphisms.

Example 3.1.2. A homomorphism of groups $p : G \to H$ gives rise to a functor $p \colon \mathcal{S} \to \mathcal{C}$ as in Example 2.1.5. This functor $p \colon \mathcal{S} \to \mathcal{C}$ is fibred in groupoids if and only if p is surjective. The fibre category $\mathcal{S}_U$ over the (unique) object $U \in \mathrm{Ob}(\mathcal{C})$ is the category associated to the kernel of p as in Example 2.1.2.

Suppose that for every $f : V \to U$ and $x \in \mathrm{Ob}(\mathcal{S}_U)$ as in the first condition we choose a lift $f^*x \to x$ of f; this is possible by the axiom of choice. For every morphism $\phi : x \to x'$ in $\mathcal{S}_U$ there is a unique morphism $f^*\phi : f^*x \to f^*x'$ in $\mathcal{S}_V$ such that

$$\begin{tikzcd}
f^*x \arrow[r, "f^*\phi"] \arrow[d] & f^*x' \arrow[d] \\
x \arrow[r, "\phi"] & x'
\end{tikzcd}$$

commutes. Again uniqueness of this arrow guarantees that f^* is a functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$.

Lemma 3.1.3. *If $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids then all fibre categories are groupoids. Choose functors f^* as above. Then for any pair of composable morphisms $f : V \to U$, $g : U \to W$ there is a unique isomorphism of functors $\mathcal{S}_W \to \mathcal{S}_V$*

$$t : g^* f^* \to (g \circ f)^*$$

such that for every $y \in Ob(\mathcal{S}_W)$ the following diagram commutes

$$\begin{tikzcd}
f^*g^*y \arrow[r] \arrow[d, "t_y"] & g^*y \arrow[d] \\
(f \circ g)^*y \arrow[r] & y
\end{tikzcd} \tag{3.1.1}$$

Proof. To show all fibre categories $\mathcal{S}_U$ for $U \in \mathrm{Ob}(\mathcal{C})$ are groupoids, we must exhibit for every $f : y \to x$ in $\mathcal{S}_U$ an inverse morphism. The diagram on the left (in $\mathcal{S}_U$) is mapped by p to the diagram on the right:

$$\begin{tikzcd}
y \arrow[r, "f"] & x \\
x \arrow[u, dashed] \arrow[ur, "id_x"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "id_U"] & U \\
U \arrow[u, dashed] \arrow[ur, "id_U"'] &
\end{tikzcd}$$

Since only id_U makes the diagram on the right commute, there is a unique $g : x \to y$ making the diagram on the left commute, so $fg = id_x$. By a similar argument there is a unique $h : y \to x$ so that $gh = id_y$. Then $fgh = f : y \to x$. We have $fg = id_x$, so $h = f$.

Now let $y \in \mathrm{Ob}(\mathcal{S}_W)$ and consider the diagram

(3.1.2)
$$
\begin{tikzcd}
f^*g^*y \arrow[r] \arrow[d, dashed, "t_y"'] & g^*y \arrow[r] & y \\
(g \circ f)^*y \arrow[urr] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, dashed, "\mathrm{id}_V"'] & U \arrow[r, "g"] & W \\
V \arrow[urr, "g \circ f"'] & &
\end{tikzcd}
$$

The morphism $t_y \colon f^*g^*y \to (g \circ f)^*y$ is the unique lift of of id_V making 3.1.2 (resp. 3.1.1) commute. If $\phi \colon y' \to y$ is a morphism in $\mathcal{S}_W$ the compositions $(f^*g^*\phi) \circ t_y$ and $((g \circ f)^*\phi) \circ t_{y'}$ are both lifts of id_V, so are equal making t is a transformation of functors. Essentially the same construction applies to give the inverse transformation t^{-1}, so t is an isomorphism. □

Conversely, given $p : \mathcal{S} \to \mathcal{C}$, we can ask: if the fibre category $\mathcal{S}_U$ is a groupoid for all $U \in \mathrm{Ob}(\mathcal{C})$, must $\mathcal{S}$ be fibred in groupoids over $\mathcal{C}$? We can see the answer is no as follows. Start with a category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$. Altering the morphisms in $\mathcal{S}$ which do not map to the identity morphism on some object does not alter the categories $\mathcal{S}_U$. Hence we can violate the existence and uniqueness conditions on lifts. One example is the functor from Example 3.1.2 when $G \to H$ is not surjective. Here is another example.

Example 3.1.4. Let $\mathrm{Ob}(\mathcal{C}) = \{A, B, T\}$ and $\mathrm{Mor}_{\mathcal{C}}(A, B) = \{f\}$, $\mathrm{Mor}_{\mathcal{C}}(B, T) = \{g\}$, $\mathrm{Mor}_{\mathcal{C}}(A, T) = \{h\} = \{gf\}$, plus the identity morphism for each object. See the diagram below for a picture of this category. Now let $\mathrm{Ob}(\mathcal{S}) = \{A', B', T'\}$ and $\mathrm{Mor}_{\mathcal{S}}(A', B') = \emptyset$, $\mathrm{Mor}_{\mathcal{S}}(B', T') = \{g'\}$, $\mathrm{Mor}_{\mathcal{S}}(A', T') = \{h'\}$, plus the identity morphisms. The functor $p : \mathcal{S} \to \mathcal{C}$ is obvious. Then for every $U \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{S}_U$ is the category with one object and the identity morphism on that object, so a groupoid, but the morphism $f : A \to B$ cannot be lifted. Similarly, if we declare $\mathrm{Mor}_{\mathcal{S}}(A', B') = \{f'_1, f'_2\}$ and $\mathrm{Mor}_{\mathcal{S}}(A', T') = \{h'\} = \{g'f'_1\} = \{g'f'_2\}$, then the fibre categories are the same and $f : A \to B$ in the diagram below has two lifts.

$$
\begin{tikzcd}
B' \arrow[r, "g'"] & T' \\
A' \arrow[u, dashed, "??"] \arrow[ur, "h'"'] &
\end{tikzcd}
\quad \textit{above} \quad
\begin{tikzcd}
B \arrow[r, "g"] & T \\
A \arrow[u, "f"] \arrow[ur, "gf=h"'] &
\end{tikzcd}
$$

Later we would like to make assertions such as "any category fibred in groupoids over $\mathcal{C}$ is equivalent to a split one", or "any category fibred in groupoids whose fibre categories are setlike is equivalent to a category fibred in sets". The notion of equivalence depends on the 2-category we are working with. To make sure that everybody knows what we are talking about we define the 2-category of categories over $\mathcal{C}$.

Definition 3.1.5. The 2-category of categories over $\mathcal{C}$ is defined as follows. Its objects will be functors $p : \mathcal{S} \to \mathcal{C}$ (belonging to some set, see Sets, Section 3). Its 1-morphisms will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$, and its 2-morphisms $t : G \to H$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

The 2-category of categories fibred in groupoids over $\mathcal{C}$ is the full sub-2-category of this 2-category whose objects are categories fibred in groupoids.

Lemma 3.1.6. *Let $p\colon \mathcal{S} \to \mathcal{C}$ and $p'\colon \mathcal{S}' \to \mathcal{C}$ be categories fibred in groupoids, and suppose that $G\colon \mathcal{S} \to \mathcal{S}'$ is a functor over $\mathcal{C}$. Then G is fully faithful (resp. an equivalence) if and only if for each $U \in Ob(\mathcal{C})$ the induced functor $G_U\colon \mathcal{S}_U \to \mathcal{S}'_U$ is fully faithful (resp. an equivalence).*

Proof. Clearly if G is fully faithful (resp. an equivalence) then so is G_U. So suppose that G_U is fully faithful for all $U \in \mathrm{Ob}(\mathcal{C})$. To show that G is fully faithful we have to show for any objects $x, y \in \mathrm{Ob}(\mathcal{S})$ that G induces a bijection between $\mathrm{Mor}_{\mathcal{S}}(x, y)$ and $\mathrm{Mor}_{\mathcal{S}'}(G(x), G(y))$. To this end let $\phi'\colon G(x) \to G(y)$ and set $U = p(x)$ and $V = p(y)$. As $\mathcal{S}$ is fibred in groupoids there is a lift $z \to y$ of $p'(\phi')$ in $\mathcal{S}$, and any morphisms $x \to y$ factors uniquely as $x \to z \to y$, where the map $x \to z$ lifts id_U, as in the following diagram

$$\begin{tikzcd} x \arrow[d, dashed] \arrow[dr] & \\ z \arrow[r, "\psi"] \arrow[d] & y \arrow[d] \\ U \arrow[r, "p'(\phi')"] & V \end{tikzcd}$$

Now in $\mathcal{S}'$, $G(\psi)\colon G(z) \to G(y)$ is the pullback of $G(y)$, so any morphism $G(x) \to G(y)$ factors uniquely as $G(x) \to G(z) \to G(y)$, where again the map $G(x) \to G(z)$ lifts id_U. Since G_U induces a bijection between $\mathrm{Mor}_{\mathcal{S}_U}(x, z)$ and $\mathrm{Mor}_{\mathcal{S}'_U}(G(x), G(z))$ we get that G induces a bijection between $\mathrm{Mor}_{\mathcal{S}}(x, y)$ and $\mathrm{Mor}_{\mathcal{S}'}(G(x), G(y))$, hence G is fully faithful.

Finally suppose for all G_U is an equivalence for all U, so it is fully faithful and essentially surjective. We have seen this implies G is fully faithful, and thus to prove it is an equivalence we have to prove that it is essentially surjective. This is clear, for if $z' \in \mathrm{Ob}(\mathcal{S}')$ then $z' \in \mathrm{Ob}(\mathcal{S}'_U)$ where $U = p'(z')$. Since G_U is essentially surjective we know that z' is isomorphic, in $\mathcal{S}'_U$, to an object of the form $G_U(z)$ for some $z \in \mathrm{Ob}(\mathcal{S}_U)$. But morphisms in $\mathcal{S}'_U$ are morphisms in $\mathcal{S}'$ and hence z' is isomorphic to $G(z)$ in $\mathcal{S}'$. □

Lemma 3.1.7. *The 2-category of categories over $\mathcal{C}$ has 2-fibre products. Suppose that $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ are morphisms of categories over $\mathcal{C}$. An explicit 2-fibre product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by the following description*

(1) *an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is a quadruple (U, x, y, f), where $U \in Ob(\mathcal{C})$, $x \in Ob(\mathcal{X}_U)$, $y \in Ob(\mathcal{Y}_U)$, and $f : F(x) \to G(y)$ is an isomorphism in $\mathcal{S}_U$,*

(2) *a morphism $(U, x, y, f) \to (U', x', y', f')$ is given by a pair (a, b), where $a : x \to x'$ is a morphism in $\mathcal{X}$, and $b : y \to y'$ is a morphism in $\mathcal{Y}$ such that (1) a and b induced the same morphism $U \to U'$, and (2) the diagram*

$$\begin{tikzcd} F(A) \arrow[r, "f"] \arrow[d, "F(a)"] & G(B) \arrow[d, "G(b)"] \\ F(A') \arrow[r, "f'"] & G(B') \end{tikzcd}$$

is commutative.

The functors $p : \mathcal{X} \times_\mathcal{S} \mathcal{Y} \to \mathcal{X}$ and $q : \mathcal{X} \times_\mathcal{S} \mathcal{Y} \to \mathcal{Y}$ are the forgetfull functors in this case. The transformation $\psi : F \circ p \to G \circ q$ is given on the object $\xi = (U, x, y, f)$ by $\psi_\xi = f : F(p(\xi)) = F(x) \to G(y) = G(q(\xi))$.

Proof. Let us check the universal property: let $p_W : \mathcal{W} \to \mathcal{C}$ be a category over $\mathcal{C}$, let $X : \mathcal{W} \to \mathcal{X}$ and $Y : \mathcal{W} \to \mathcal{Y}$ be functors over $\mathcal{C}$, and let $t : F \circ X \to G \circ Y$ be an isomorphism of functors. The desired functor $\gamma : \mathcal{W} \to \mathcal{A} \times_\mathcal{C} \mathcal{B}$ is given by $W \mapsto (p_W(W), X(W), Y(W), t_W)$. What else could it be? (A meta-argument for uniqueness.) FIXME: write this out. $\square$

Lemma 3.1.8. *In the situation of the lemma above, if $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{S}$ are fibred in groupoids over $\mathcal{C}$, then so is $\mathcal{X} \times_\mathcal{S} \mathcal{Y}$. In particular the 2-category of categories fibred in groupoids over $\mathcal{C}$ has 2-fibre products (and they are described as above).*

Proof. FIXME. $\square$

Subsection 3.2. **Categories fibred in sets.** Let us call a category setlike if it is a groupoid where every object has exactly one automorphism: the identity. If C is a set with an equivalence relation $\sim$, then we can make a setlike category $\mathcal{C}$ as follows: $\mathrm{Ob}(\mathcal{C}) = C$ and $\mathrm{Mor}_\mathcal{C}(x, y) = \emptyset$ unless $x \sim y$ in which case we set $\mathrm{Mor}_\mathcal{C}(x, y) = \{1\}$. Transitivity of $\sim$ means that we can compose morphisms. Conversely any setlike category defines an equivalence relation on its objects (isomorphism) such that you recover the category (up to unique isomorphism – not equivalence) from the procedure just described. This is why these categories are sometimes simply called equivalence relations.

A category is called discrete if the only morphisms are the identity morphisms. Sometimes discrete categories are called sets (reasons as above). Discrete categories are setlike. For any setlike category $\mathcal{C}$ there is a canonical procedure to make a discrete category equivalent to it, namely one replaces $\mathrm{Ob}(\mathcal{C})$ by the set of isomorphism classes, and adds identity morphisms.

Definition 3.2.1. A category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$ is said to be a category fibred in sets if all fibre categories are discrete.

We discuss briefly the relationship between categories fibred in sets and presheaves (see Sites, Definition 2.1.5). Suppose that $p : \mathcal{S} \to \mathcal{C}$ is fibred in sets. Let $f : V \to U$ be a morphism in $\mathcal{C}$ and let $x \in \mathrm{Ob}(\mathcal{S}_U)$. Then there is exactly one choice for the object f^*x. Thus we see that $(f \circ g)^*x = g^*(f^*x)$ for f, g as in Lemma 3.1.3. It follows that we may think of the assigments $U \mapsto \mathrm{Ob}(\mathcal{S}_U)$ and $f \mapsto f^*$ as a presheaf on $\mathcal{C}$.

Conversely, given a presheaf of sets $F : \mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets}$ we can construct a category $\mathcal{S}_F$ fibred in sets over $\mathcal{C}$ by taking as fibre category $\mathcal{S}_{F,U}$ the discrete category whose underlying set is $F(U)$. This is explained more generally, and in more detail in Example 3.3.1 below. Also, here is an important example.

Example 3.2.2. In this example $F = h_X = \mathrm{Mor}(-, X)$ for some $X \in \mathrm{Ob}(\mathcal{C})$ (see Example 2.1.8). In other words, F is a representable presheaf. Since $\mathcal{S}_{F,U}$ is the discrete category whose objects are the morphisms from U into X it follows that $\mathcal{S}_F \to \mathcal{C}$ is the functor denoted $\mathcal{C}/X \to \mathcal{C}$ from Example 2.1.6. FIXME. Improve formulation.

For this reason it is tempting to define a "representable" object in the 2-category of categories fibred in groupoids to be a category fibred in sets whose associated presheaf is representable. However, this is would not be a good definition since we prefer to have a notion wich is invariant under equivalences. Thus we consider first which categories in groupoids are equivalent to categories fibred in sets.

Lemma 3.2.3. *Suppose that $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids all of whose fibre categories $\mathcal{S}_U$ are setlike. Then there exists a category fibred in sets $p' : \mathcal{S}' \to \mathcal{C}$ and an equivalence $can : \mathcal{S} \to \mathcal{S}'$ of categories over $\mathcal{C}$. The 1-morphism $\mathcal{S} \to \mathcal{S}'$ is unique up to a unique 2-morphism. It further has the property that*

$$Ob(\mathcal{S}_U) \longrightarrow Ob(\mathcal{S}'_U)$$

(induced by can) identifies the RHS with ismorphism classes of the LHS for all $U \in Ob(\mathcal{C})$. The 1-morphism $\mathcal{S} \to \mathcal{S}'$ is unique up to a unique 2-morphism.

Conversely, any category fibred in groupoids over $\mathcal{C}$ which is equivalent (as a category over $\mathcal{C}$) to a category fibred in sets, has setlike fibre categories.

Proof. An object of the category $\mathcal{S}'$ will be a pair (U, ξ), where $U \in \mathrm{Ob}(\mathcal{C})$ and ξ is an isomorphism class of objects of $\mathcal{S}_U$. A morphism $(U, \xi) \to (V, \psi)$ is given by a morphism $x \to y$, where $x \in \xi$ and $y \in \psi$. Here we identify two morphisms $x \to y$ and $x' \to y'$ if they induce the same morphism $U \to V$, and if for some choices of isomorphisms $x \to x'$ in $\mathcal{S}_U$ and $y \to y'$ in $\mathcal{S}_V$ the compositions $x \to x' \to y'$ and $x \to y \to y'$ agree. By construction there are surjective maps on objects and morphisms from $\mathcal{S} \to \mathcal{S}'$. We define composition of morphisms in $\mathcal{S}'$ to be the unique law that turns $\mathcal{S} \to \mathcal{S}'$ into a functor. FIXME: check this is well-defined.

By construction the rule $(U, \xi) \mapsto U$ is a functor. FIXME: check this and the other properties. □

With this lemma in hand it is easy to recognize those categories over $\mathcal{C}$ which are equivalent to a category fibred in sets. Thus we now make the following definition.

Definition 3.2.4. A category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$ is called representable, if the following conditions are satisfied:

(1) all fibre categories $\mathcal{S}_U$ are setlike, and
(2) the presheaf $U \mapsto \mathrm{Ob}(\mathcal{S}_U)/\cong$ is representable.

In this case, by Lemma 3.2.3 the category $\mathcal{S}'$ is isomorphic to $\mathcal{C}/X$ over $\mathcal{C}$. As usual, by the Yoneda lemma the pair (X, j), where j is the equivalence $j : \mathcal{S} \to \mathcal{C}/X$ is uniquely determined up to isomorphism.

Lemma 3.2.5. *The 2-category of categories fibred in sets over $\mathcal{C}$ has 2-fibre products. More precisely, the 2-fibre product described in Lemma 3.1.7 returns a category fibred in sets if one starts out with such. A similar result holds for categories fibred in groupoids all of whose fibre categories are setlike.*

Proof. FIXME. □

Subsection 3.3. Presheaves of groupoids. In this subsection we compare the notion of categories fibred in groupoids with the closely related notion of a "presheaf of groupoids". The basic construction is explained in the following example.

Example 3.3.1. Suppose that $F : \mathcal{C} \to \text{Groupoids}$ is a contravariant functor to the category of groupoids (see Remark 2.1.4 and Remark 2.2.5). For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. From this we can construct a category fibred in groupoids over $\mathcal{C}$ as follows. Define

$$\text{Ob}(\mathcal{S}) = \{(U, x) \mid U \in \text{Ob}(\mathcal{C}), x \in \text{Ob}(F(U)\}.$$

For $(U, x), (V, y) \in \text{Ob}(\mathcal{S})$ we define

$$\text{Mor}_{\mathcal{S}}((V, y), (U, x)) = \{(f, \phi) \mid f \in \text{Mor}_{\mathcal{C}}(V, U), \phi \in \text{Mor}_{F(V)}(y, f^*x)\}.$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$ (by definition of a functor into a 2-category). Namely, we define the composition of $\psi : z \to g^*y$ and $\phi : y \to f^*x$ to be $g^*(\phi) \circ \psi$. It is clear what the functor $p : \mathcal{S} \to \mathcal{C}$ is. The condition that $F(U)$ is a groupoid for every U guarantees that $\mathcal{S}$ is fibred in groupoids over $\mathcal{C}$. Lifts of morphisms exist: given $f : V \to U$ in $\mathcal{C}$ and (U, x) a lift of U, then $(f, id_{f^*x}) : (V, f^*x) \to (U, x)$ is a lift of f. Uniqueness means h in the diagram on the left determines (h, ν) on the right:

$$\begin{tikzcd} V \arrow[r, "f"] & U \\ W \arrow[u, dashed, "h"] \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} (V, y) \arrow[r, "{(f,\phi)}"] & (U, x) \\ (W, z) \arrow[u, dashed, "{(h,\nu)}"] \arrow[ur, "{(g,\psi)}"'] & \end{tikzcd}$$

Then $\nu = (h^*\phi)^{-1} \circ \psi$ and the uniqueness of inverses guarantees this is the only lift making the diagram commute.

We will write $\mathcal{S}_F \to \mathcal{C}$ for the resulting functor if we want to indicate the dependence on F. Because we can think of objects of $\mathcal{S}_F$ as pairs (U, x), we sometimes say $\mathcal{S}_F$ is a *split* category fibred in groupoids.

Lemma 3.3.2. *Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. There exists a functor $F : \mathcal{C} \to$ Groupoids such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ over $\mathcal{C}$. In other words, every category fibred in groupoids is equivalent to a split one.*

Proof. We construct a new category $\mathcal{S}'$ as follows. First we choose pullback functors $g^* : \mathcal{S}_V \to \mathcal{S}_{V'}$ for any morphism $g : V' \to V$ of $\mathcal{C}$. (We can do this since $\mathcal{S}$, $\mathcal{C}$ are sets. FIXME: We can do this proof without choosing these as well.) The objects of $\mathcal{S}'$ are pairs (x, f) consisting of a morphism $f : V \to U$ of $\mathcal{C}$ and an object x of $\mathcal{S}$ over U, i.e., $x \in \text{Ob}(\mathcal{S}_U)$. The functor $p' : \mathcal{S}' \to \mathcal{C}$ will map the pair (x, f) to the source of the morphism f, in other words $p'(x, f : V \to U) = V$. A morphism $\varphi : (x_1, f_1 : V_1 \to U_1) \to (x_2, f_2 : V_2 \to U_2)$ is given by a pair (φ, g) consisting of a morphism $g : V_1 \to V_2$ and a morphism $\varphi : f_1^*x_1 \to f_2^*x_2$ with $p(\varphi) = g$. It is no problem to define the composition law: $(\varphi, g) \circ (\psi, h) = (\varphi \circ \psi, g \circ h)$ for any pair of composable morphisms. There is a natural functor $\mathcal{S} \to \mathcal{S}'$ which simply maps x over U to the pair (x, id_x).

FIXME. We need to check that p' makes $\mathcal{S}'$ into a category fibred in groupoids over $\mathcal{C}$, and we need to check that $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories over $\mathcal{C}$ (hopefully the lemma above helps!).

Finally, we can define pullback functors on $\mathcal{S}'$ by setting $g^*(x, f) = (x, f \circ g)$ on objects if $g : V' \to V$ and $f : V \to U$. On morphisms $(\varphi, \text{id}_V) : (x_1, f_1) \to (x_2, f_2)$ between morphisms in $\mathcal{S}'_V$ we set $g^*(\varphi, \text{id}_V) = (g^*\varphi, \text{id}_{V'})$ where we use the unique identifications $g^*f_i^*x_i = (f_i \circ g)^*x_i$ from Lemma 3.1.3 to think of $g^*\varphi$ as a morphism from $(f_1 \circ g)^* x_1$ to $(f_2 \circ g)^* x_2$. Clearly, these pullback functors g^* have the property that $g_1^* \circ g_2^* = (g_2 \circ g_1)^*$, in other words $\mathcal{S}'$ is split as desired. ☐

Alternate proof. We define a contravariant functor F from $\mathcal{C}$ to the category of groupoids as follows: for $U \in \mathrm{Ob}(\mathcal{C})$ set $F(U) = \mathrm{Mor}(\mathcal{S}/U, \mathcal{S})$ to be the set of base preserving natural transformations. If $f : U \to V$ the induced functor $\mathcal{S}/U \to \mathcal{S}/V$ induces the morphism $F(f) \colon F(V) \to F(U)$. Clearly F is a functor, and we will see below that it is a functor into groupoids. Let $\mathcal{S}'$ be the associated category fibred in groupoids from Example 3.3.1.

There is an obvious functor $G \colon \mathcal{S}' \to \mathcal{S}$ over $\mathcal{C}$ given by taking the pair (U, x), where $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(U)$, to $x(U \xrightarrow{\mathrm{id}_U} U) \in \mathcal{S}$. Now Lemma 3.3.3 implies that for each U,

$$G_U : \mathcal{S}'_U = F(U) = \mathrm{Mor}(\mathcal{C}/U, \mathcal{S}) \to \mathcal{S}_U$$

is an equivalence, and thus G equivalence between $\mathcal{S}$ and $\mathcal{S}'$ by Lemma 3.1.6. ☐

Lemma 3.3.3. *Let $\mathcal{S} \to \mathcal{C}$ be fibred in groupoids. Then for any $U \in Ob(\mathcal{C})$ the functor*

$$G \colon Mor(\mathcal{C}/U, \mathcal{S}) \to \mathcal{S}_U$$

given by $G(x) = x(U \xrightarrow{id_U} U)$ is an equivalence.

FIXME: Do we have notation for base preserving transformations already? Say what G does on arrows.

Proof. We define a functor $H \colon \mathcal{S}_U \to \mathrm{Mor}(\mathcal{C}/U, \mathcal{S})$ as follows. Given $x \in \mathrm{Ob}(\mathcal{S}_U)$ and $f \colon X \to U$ set $H(x)(f) = f^* x$. (FIXME: say what this does on arrows and prove this gives an equivalence). ☐

Biographical notes: Parts of this have been taken from Vistoli's notes [Vis].

To continue reading,

(1) visit the next section: Sites, Section 1, or
(2) go back to the table of contents: index.html#contents.

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